Minimization on Riemannian manifolds and the application to Cosserat models

Wolfgang Müller and Christian Wieners

Faculty of Mathematics
Universität Karlsruhe (TH)

15th April 2005

Infinitesimal Cosserat Model

Compute displacement $\mathbf{u}$ and infinitesimal rotations $\mathrm{id} + \overline{A}$ with $\overline{A}^T = -\overline{A}$, such that

$$\int W(\mathbf{u}, \overline{A}) - \ell(\mathbf{u}) = \min$$

where the stored energy is

$$\begin{aligned} W(\mathbf{u}, \overline{A}) &= \mu \, \| \mathrm{sym}(\nabla \mathbf{u}) \|^2 + \frac{\lambda}{2} \, \mathrm{tr}(\mathrm{sym}(\nabla \mathbf{u}))^2 \\ &\quad + \mu_c \, \| \mathrm{skew}(\nabla \mathbf{u}) - \overline{A} \|^2 + \mu L_c^2 \, \| \nabla \overline{A} \|^2 . \end{aligned}$$

Finite Cosserat Model

Compute displacement $\mathbf{u}$ and rotations $\overline{R}$ with $\overline{R}^T\overline{R} = \mathrm{id}$, such that

$$\int W(\mathbf{u}, \overline{R}) - \ell(\mathbf{u}) = \min$$

where the stored energy is

$$\begin{aligned} W(\mathbf{u}, \overline{R}) &= \mu \, \| \operatorname{sym}(\overline{R}^T F - \mathrm{id}) \|^2 + \frac{\lambda}{2} \operatorname{tr}(\operatorname{sym}(\overline{R}^T F - \mathrm{id}))^2 \\ &\quad + \mu_c \, \| \operatorname{skew}(\overline{R}^T F) \|^2 + \mu \left[1 + L_c^2 \, \| \overline{R}^T D \overline{R} \|^2 \right]^{q/2} \end{aligned}$$

depending on the deformation gradient $F = \mathrm{id} + \nabla u$.

Simplified Cosserat Model

For given $\mathbf{u}\colon [0,1] \longrightarrow \mathbf{R}^3$ compute rotations $\overline{R}$ with $\overline{R}^T\overline{R} = \mathrm{id}$, such that

$$f(\overline{R}) := \int W(\mathbf{u}, \overline{R}) - \ell(\mathbf{u}) = \min$$

where for $F = \mathrm{id} + \nabla u$

$$\begin{aligned} W(\mathbf{u}, \overline{R}) &= \mu \, \|\operatorname{sym}(\overline{R}^T F - \mathrm{id})\|^2 + \frac{\lambda}{2} \operatorname{tr}(\operatorname{sym}(\overline{R}^T F - \mathrm{id}))^2 \\ &\quad + \mu_c \, \|\operatorname{skew}(\overline{R}^T F)\|^2 + \mu \left[1 + L_c^2 \, \|\overline{R}^T D\overline{R}\|^2\right]^{q/2}, \end{aligned}$$

Goal: Minimize $f(\overline{R})$ on the manifold $M = \{\overline{R}\colon \overline{R}^T\overline{R} = \mathrm{id}\}$.

The Newton method for minimization on manifolds

Let M be a Riemannian manifold with metric $\langle\cdot,\cdot\rangle$, and let $f\colon M \to \mathbf{R}$ be a smooth functional on M. We consider the minimization problem

$$p \in M\colon \qquad f(p) = \min!$$

For the definition of the Newton algorithm, we linearize the problem in the tangent space T_pM. Therefore, for a vector $v \in T_pM$, let $\gamma_v\colon [0,T] \longrightarrow M$ be the geodesic curve starting at $p = \gamma_v(0)$ with $\dot\gamma_v(0) = v$. This defines $\exp_p(v) := \gamma_v(1)$ on

$$\mathcal{D}_p = \{v \in T_pM\colon \gamma_v \text{ is defined at least for } t \in [0,1]\}.$$

Furthermore, we set $\mathcal{V}_p = \exp_p(\mathcal{D}_p) \subset M$.

The Newton method for minimization on manifolds

Starting at $p_0 \in M$, in every step $n = 0, 1, 2, \ldots$ we consider the local problem

$$v \in \mathcal{D}_{p_n}: \qquad f(\exp_{p_n}(v)) = \min!$$

For the computation of the Newton increment, we define $f_{p_n}\colon \mathcal{D}_{p_n} \longrightarrow \mathbf{R}$ by

$$f_{p_n}(v) := f(\exp_{p_n}(v)),$$

consider the Taylor expansion

$$f_{p_n}(v) = f_{p_n}^{(0)} + f_{p_n}^{(1)}[v] + \frac{1}{2} f_{p_n}^{(2)}[v, v] + \cdots$$

and solve the linear problem

$$v_n \in T_{p_n}M: \qquad f_{p_n}^{(2)}[v_n, w] = -f_{p_n}^{(1)}[w] \,, \qquad w \in T_{p_n}M \,.$$

If $v_n \in \mathcal{D}_{p_n}$, this gives the new iterate $p_{n+1} = \exp_{p_n}(v_n)$.

The theorem of Newton-Kantorovich on manifolds

Assume that for $p_0 \in M$ holds:

a) $|\langle \operatorname{grad} f(p_0), w\rangle_{p_0}| \le \eta \, \|w\|_{p_0}$ for $w \in T_{p_0}M$;

b) $\langle \operatorname{Hess} f(p_0)w, w\rangle_{p_0} \ge \beta \, \|w\|_{p_0}^2$ for $w \in T_{p_0}M$;

c) $K \subset \mathcal{V}_p$ for $p \in K$, where $K := \{p \in M \colon d(p, p_0) \le r\}$ and $r = 2\eta/\beta$;

d) $\operatorname{Hess} f$ is locally Lipschitz in the following sense: for any geodesic curve $\gamma\colon [a, b] \longrightarrow K$ with $\gamma(a) = p$ and $\gamma(b) = q$ we have for all $w, w' \in T_pM$

$$\Big|\langle \operatorname{Hess} f(q) P_{\gamma,a,b} w, P_{\gamma,a,b} w'\rangle_q - \langle \operatorname{Hess} f(p) w, w'\rangle_p\Big| \le \lambda \, \|w\|_p \|w'\|_p \, d(p, q) ;$$

e) $2\eta\lambda \le \beta^2$.

The theorem of Newton-Kantorovich on manifolds

Then, we have: the Newton sequence p_n builds a well defined Cauchy sequence in K with limit $p^* \in K$, and we have the estimate

$$d(p^*, p_0) \leq \frac{2\eta}{\beta + \sqrt{\beta^2 - 2\eta\lambda}} \leq r$$

and

$$d(p^*, p_n) \leq \frac{\beta}{\lambda}\Big(\frac{2\lambda\eta}{\beta^2}\Big)^{2^n} \ .$$

Moreover, we have

$$\langle \mathrm{Hess}\, f(p^*)w, w\rangle_{p^*} \geq \Big(\beta - \lambda\, d(p^*, p_0)\Big)\|w\|^2_{p^*} \quad \text{for} \quad w \in T_{p^*}M \ .$$

If, in addition, $2\eta\lambda < \beta^2$, Newton's method converges quadratically and p^* is a isolated local minimum of f in K.

Application to the manifold $SO(3)^N$

We compute $R^0 = R_a, R^1, ..., R^N = R_b$ at prescribed points $\varphi(z_0), ..., \varphi(z_N) \in \mathbf{R}^3$ with $a = z_0 < z_1 < \cdots < z_N = b$. Replacing $\partial_z R$ by a difference quotient in the interior respectively a simple one at the boundary. Using trapezoidal rule, we obtain the minimization problem

$$f(R^1, ..., R^{N-1}) = \min!$$

depending on R^0, R^N and $F^0 = \nabla\varphi(z_0), ..., F^N = \nabla\varphi(z_N)$:

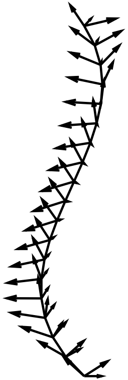
$$f(R^1, ..., R^{N-1}) = \frac{\lambda}{2} \sum_{m=1}^{N-1} h_{m-1/2} \big(\mathrm{tr}((R^m)^T F^m) - 3 \big)^2$$

$$+ \mu \sum_{m=1}^{N-1} h_{m-1/2} \, \| \mathrm{sym}((R^m)^T F^m - I) \|^2 + \mu L_c^2 \sum_{m=0}^{N-1} \frac{1}{h_m} \, \| R^{m+1} - R^m \|^2$$

$(h_m = z_{m+1} - z_m, \; h_{m-1/2} = (h_{m-1} + h_m)/2)$.

Numerical experiment (Globalized Newton method)

Newton defect	5e02	...	1e-01	1e-05	1e-12
number of steps	0	...	54	55	56



Summary, outlook and future work

Already realized:

- Finite element implementation for linear infinitesimal Cosserat models in 2D and 3D
- Nonlinear Newton analysis for minimization problems on Riemannian manifolds (e. g., Cosserat rotations)

Next steps:

- Multigrid methods for Cosserat models
- Elasto-plastic Cosserat models
- Full nonlinear Cosserat models