Department of Mathematics
Institute for Applied and
Numerical Mathematics

JProf. Dr. Katharina Schratz
Dipl-Math. techn. Patrick Krämer

Aspects of Numerical Time Integration — Exercise Sheet 01 April 16, 2015

On this exercise sheet we want to practice the implementation of simple time integration schemes in MATLAB such as the explicit and implicit Euler method. For $T > 0$ we consider the ODE

$$y'(t) = f(y(t)), \quad y(0) = y_0 \in \mathbb{R}^d, \qquad f : \mathbb{R}^d \to \mathbb{R}^d \text{ smooth}, \qquad t \in [0, T] \tag{1}$$

and discretize our time interval such that $t_n = nh$, $n = 0, 1, 2, \ldots, N$, $t_N = T$, with $h \in (0, 1)$ a small **time step size**. A numerical time integration scheme for (1) should satisfy $y_n \approx y(t_n)$, $\forall t_n < T$, i.e. the numerical method should approximate the exact solution of the ODE.

We define the **flow** φ of a differential equation as the mapping, which maps an initial value to the exact solution at some time $t \in [0, T]$. Denote by $\varphi_f^t(y_0)$ flow of (1), i.e. $\varphi_f^t(y_0) = y(t)$.

We say a numerical method $\Phi^h(y_n) := y_{n+1}$ is **consistent** of/ has a **local error** of order p if the local error satisfies

$$\left\| \Phi^h\Big(y(t_n)\Big) - \varphi_f^h\Big(y(t_n)\Big) \right\| \le Ch^{p+1}.$$

Exercise 1:

a) Show that the explicit Euler method

$$y_{n+1} = y_n + hf(y_n)$$

is consistent of order $p = 1$.

b) Consider the first order ODE

$$y'(t) = i\,|y(t)|^2 y(t), \qquad y(0) = y_0.$$

• Can you give the **exact solution** $y(t)$ explicitly?

Implement the explicit Euler method applied to this ODE with $y_0 = \frac{1-2i}{\sqrt{5}}$ in MATLAB with time step size $h = 0.1$ on the time interval $t \in [0, T = 10]$ and compare the numerical solution with the exact solution, i.e. plot the evolution of

$$|y_n - y(t_n)|$$

over all times t_n.

• What can you see?

• Can you give an explanation for your observation?

c) Create an order plot and show that we find the order of consistency $p = 1$ also numerically, i.e. for various time step sizes $h = 2^{-j}, j \in \{3, 4, \ldots, 10\}$ run the simulation with the explicit Euler method and `loglog`-plot the maximal error

$$\max_{t_n \in [0,T]} |y_n - y(t_n)|$$

for each value of h against the corresponding time step size h. You should see a line with slope 1.

Discussion in the problem class tuesday 3:45 pm, in room 3.069 in the Kollegiengebäude Mathematik 20.30.

♣ : Please try to do exercises marked with ♣ at home.