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Aspects of Numerical Time Integration — Exercise Sheet 02 April 24, 2015

On this exercise sheet we want to recall the definition and the practical implementation of splitting methods.

Let $T > 0$. We consider the ODE

$$y'(t) = f^{[1]}(y(t)) + f^{[2]}(y(t)) := f(y(t)), \quad y(0) = y_0 \in \mathbb{R}^d, \quad f^{[1]}, f^{[2]} \text{ smooth} \tag{1}$$

on the time interval $t \in [0, T]$ with exact flow $\varphi_f^t(y_0)$.

Very often the subproblems

$$y'(t) = f^{[1]}(y(t)), \quad y(0) = v_0 \in \mathbb{R}^d, \tag{S1}$$

$$y'(t) = f^{[2]}(y(t)), \quad y(0) = w_0 \in \mathbb{R}^d \tag{S2}$$

with exact flows $\varphi_{f^{[1]}}^t(v_0), \varphi_{f^{[2]}}^t(w_0)$ are more easy to solve than the original problem, sometimes they are even exactly solvable. We can exploit this property by applying a splitting method such that for a small time step size $0 < \tau < 1$ we get for instance with the **Lie splitting method**

$$\Phi_L^\tau(y_0) := \varphi_{f^{[2]}}^\tau \circ \varphi_{f^{[1]}}^\tau(y_0) \approx \varphi_f^\tau(y_0),$$

and with the **Strang splitting method**

$$\Phi_S^\tau(y_0) := \varphi_{f^{[1]}}^{\tau/2} \circ \varphi_{f^{[2]}}^\tau \circ \varphi_{f^{[1]}}^{\tau/2}(y_0) \approx \varphi_f^\tau(y_0).$$

Exercise 2:

a) ♣ Let $0 < \tau < 1$. For the example of the linear case, i.e. $f(y) = Ay + By, \; A, B \in \mathbb{R}^{d\times d}$, show that the Lie splitting method Φ_L^τ is consistent of order $p = 1$ (cf. Exercise Sheet 1).

b) Consider the ODE

$$y'(t) = i\Lambda y(t) + i|y(t)|^2 y(t), \quad y(0) = y_0 \in \mathbb{R}^2, \quad t \in [0, T], \qquad \Lambda \in \mathbb{R}^{2\times 2}. \tag{2}$$

• Explain why a splitting method makes sense for this problem.

We set $y_0 = \frac{\tilde{y}_0}{\|\tilde{y}_0\|}$ with $\tilde{y}_0 = \begin{pmatrix} 1-2i \\ 3-4i \end{pmatrix}$ and $\Lambda = \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix}$.

• In MATLAB implement the **Lie splitting method** applied to (2) with time step size $\tau = 2^{-5}$ on the time interval $t \in [0, T = 10]$. Plot the modulus of the numerical solution, i.e. plot $|y_n|^2$ over all times $t_n < T$.

• In MATLAB implement the **explicit Euler method** applied to (2) with time step size $\tau = 2^{-5}$ on the time interval $t \in [0, T = 10]$. Plot the modulus of the numerical solution, i.e. plot $|y_n|^2$ over all times $t_n < T$.

For many problems, in particular for (2), it is not easy and very often also impossible to write down an explicit formula for the exact solution.

But if we want to investigate the error of a numerical method Φ^τ applied to such a problem numerically we need so called numerical **reference solutions**.

Of course a reference solution should be of better numerical quality than the solution obtained with the method Φ^τ. Hence in order to compute a reference solution we choose a numerical method which is of a higher order of consistency than Φ^τ and use a smaller time step size $\tau_R = \tau/M$.

c) In MATLAB implement the **Strang splitting method** $\Phi_S^{\tau_R}$ applied to (2) as a reference method for the methods of part b) on the interval $t \in [0, T = 1]$ with time step size $\tau_R = \tau/8$.

• Why is the Strang splitting method a suitable choice for a reference method?

• As on exercise sheet 1 plot the evolution of the error of the Lie splitting method and the explicit Euler method and create an order plot for these methods. What can you observe?

Discussion in the problem class tuesday 3:45 pm, in room 3.069 in the Kollegiengebäude Mathematik 20.30.

♣ : Please try to do exercises marked with ♣ at home.