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Aspects of Numerical Time Integration — Exercise Sheet 03 April 30, 2015

When we have to deal with partial differential equations (PDEs) instead of ODEs, then we need suitable space discretization schemes. On this exercise sheet we want to introduce the scheme of **finite differences** for spatial discretization. We study the numerical behaviour of this scheme applied to a linear wave equation

$$\partial_{tt}u(t,x) = \partial_{xx}u(t,x) - u(t,x), \quad u(0,x) = \sin(x),\ \partial_t u(0,x) = -\sqrt{2}\cos(x), \quad t \in [0,T], \quad x \in [a,b]. \tag{1}$$

Furthermore we assume periodic boundary conditions, i.e.

$$u(t,a) = u(t,b), \quad \partial_x u(t,a) = \partial_x u(t,b).$$

Exercise 3:

♣ Show that $\tilde{u}(t,x) = \sin(x - \sqrt{2}t)$ is the exact solution of (1).

If we want to solve a PDE involving time derivatives and spatial derivatives we have to choose a suitable time integration scheme as well as a suitable spatial discretization.

The spatial discretization scheme of **finite differences** for a smooth function $g(x)$ at grid points $x_j = x_0 + jh,\ j = 1,\dots,N_x,\ x_0 = a,\ x_{N_x} = b$ with $g(x_{N_x}) = g(x_0)$ and $g^j = g(x_j), j = 0,\dots,N_x - 1$ is given as

$$\partial_{xx}g(x_j) \approx \frac{1}{h^2}(g^{j-1} - 2g^j + g^{j+1}) =: g^j_{xx}.$$

Exercise 4:

a) ♣ Show that for the error of the finite difference scheme there holds

$$|\partial_{xx}g(x_j) - g^j_{xx}| \le Ch^2, \quad \text{for sufficiently smooth } g.$$

b) Find a matrix $A \in \mathbb{R}^{N_x \times N_x}$ such that

$$\partial_{xx}g(x_j) \approx (A\tilde{g})_j, \quad j = 0,\dots,N_x - 1,$$

where $\tilde{g} = [g^0, g^1, \dots, g^{N_x - 1}]^T$.

After spatial discretization with finite differences, equation (1) reduces to a problem which is only time dependent, i.e.

$$\tilde{u}''(t) = A\tilde{u}(t) - \tilde{u}(t), \quad \tilde{u}(0) = (\sin(x_j))_{j=0}^{N_x-1},\ \tilde{u}'(0) = (-\sqrt{2}\cos(x_j))_{j=0}^{N_x-1}, \quad t \in [0,T], \tag{2}$$

where $(\tilde{u}(t))^j \approx u(t,x_j)$ such that we can apply a suitable time integration scheme.

Let $0 < \tau < 1$ a small time step size and $t_n = n\tau, n = 0,1,2,\dots,N,\ t_N \le T$. Furthermore let $u_n \approx \tilde{u}(t_n)$, $v_n \approx \tilde{u}'(t_n)$ approximate the exact solution of (2). We choose the Störmer-Verlet scheme for the time integration which reads as follows.

$$\begin{aligned} v_{n+\frac{1}{2}} &= v_n + \frac{1}{2}\tau\,(Au_n - u_n) \\ u_{n+1} &= u_n + \tau v_{n+\frac{1}{2}} \\ v_{n+1} &= v_{n+1/2} + \frac{1}{2}\tau\,(Au_{n+1} - u_{n+1}) \end{aligned} \tag{SV}$$

please turn over →

Exercise 5:

a) In MATLAB implement the **Störmer-Verlet time integration method** applied to (1) on the spatial interval $x \in [a = -\pi, b = \pi]$ and on the time interval $t \in [0, T = 1]$ with time step size $\tau = 2^{-8}$ (cf. (SV)) using **finite differences** for the spatial discretization with $N_x = 8$ grid points and hence with spatial step size $h = (b - a)/N_x$.

b) Compare your numerical solution with the exact solution for various number of grid points $N_x^l = 8 \cdot 2^l, l = 0, \dots, 7$ at a **fixed** time step size $\tau = 2^{-8}$ and create an order plot to see the order m of the spatial error, i.e. you should see a line with slope k if the spatial error is of order $\mathcal{O}(h^k)$.

   Do you have an explanation for the behaviour of the line at $N_x^7 = 2^{10}$?

c) Compare your numerical solution with the exact solution for various time step sizes $\tau^l = 2^{-10} \cdot 2^l, l = 0, \dots, 6$ at a **fixed** number of grid points $N_x = 150$ and create an order plot to see the order p of the time integration error, i.e. you should see a line with slope p if the global time integration error is of order $\mathcal{O}(\tau^p)$.

   Do you have an explanation for the behaviour of the line for small values of τ?

   Now change the number of grid points to $N_x = 256$. How does the line change and how can you explain this?

Discussion in the problem class Tuesday 3:45 pm/ Wednesday 2 pm , in room 3.069 in the Kollegiengebäude Mathematik 20.30.

♣ : Please try to do exercises marked with ♣ at home.