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Aspects of Numerical Time Integration — Exercise Sheet 04 May 20, 2015

On this exercise sheet we want to get familiar with another method of spatial discretization, the so-called *spectral method*. As we will see, spectral methods work very well for periodic problems, but we have to be careful with their application to problems which are made periodic artificially for practical implementation.

Motivation and definition: (cf. literature in footnote) Let $N > 0$ and let $h = (2\pi)/N$ and $x_j = -\pi + jh,\ j = 0, \ldots, N$ a discretization of the interval $[-\pi, \pi]$. Let $u\colon \mathbb{R} \to \mathbb{C}$ smooth and 2π-periodic on $[-\pi, \pi]$, i.e. $u(-\pi) = u(\pi)$ and in particular $u(x_0) = u(x_N)$. Then the Fourier expansion of u reads $u(x) = \sum_{k\in\mathbb{Z}} \hat{u}_k e^{ikx}$.

The idea is now to use a **trigonometric interpolation polynomial** $t_N(x)$ to approximate $u(x)$ with interpolation property in the grid points x_j, i.e. $t_N(x_j) = u(x_j),\ j = 0, \ldots, N$. Let $U \in \mathbb{C}^N$ with $U_j = u(x_j), j = 0, \ldots, N-1$ and let

$$t_N(x) = \frac{1}{2N}\left(\hat{U}_{-N/2}e^{-ixN/2} + \hat{U}_{N/2}e^{ixN/2}\right) + \frac{1}{N}\sum_{k=-N/2+1}^{N/2-1} \hat{U}_k e^{ikx},$$

where $\hat{U} = \mathcal{F}_N U$ is the **discrete Fourier transform** of U.

Exercise 6:

Show that t_N satisfies the interpolation property, i.e. show that there holds $t_N(x_j) = u(x_j)$.

Hint: Exploit periodicity of the solution and the definition of the discrete Fourier transform.

On the basis of the trigonometric polynomial t_N and the discrete Fourier transform we can derive spectral differentiation methods such that for the m-th derivative of u there holds

$$u^{(m)}(x_j) \approx t_N^{(m)}(x_j) = \mathcal{F}_N^{-1}\left((i\tilde{k})^m \cdot \mathcal{F}_N U\right)_j, \quad j = 0, \ldots, N-1, \tag{1}$$

where

$$\tilde{k} = [-N/2+1, \ldots, N/2-1, \chi], \quad \chi = \begin{cases} N/2, & m \text{ even}, \\ 0, & m \text{ odd}. \end{cases}$$

The theory on spectral derivatives also works for arbitrary intervals $[a, b]$, but then we have to transform the interval $[a, b]$ to an interval of length 2π. Which **additional factor** do we then need in the coefficients $\tilde{k}$?

If we use the MATLAB built-in function `fft` and `ifft` respectively and if we set $N = 2^r, r \in \mathbb{N}$, we can compute an approximation to the m-th derivative of u in $\mathcal{O}(N \log N)$ operations. Our aim is now to compare spectral differentiation methods with the method of finite differences.

Exercise 7:

Consider $f(x) := \exp(\sin(x))$ on the interval $x \in [-\pi, \pi]$ and consider $N = 16$ grid points at first.

a) In MATLAB implement a spectral method to compute an approximation to the second derivative $f''(x)$. Plot the numerical result together with the exact derivative.

b) Add a finite difference approximation to f'' to your plot. What do you observe?

c) Create an order plot for the spatial accuracy of the spectral and the finite difference method using $N_l = 8 \cdot 2^l,\ l = 0, \ldots, 9$ grid points. Compute the corresponding errors in the approximate L^2 norm, i.e.

$$err_l = \sqrt{h_l}\, \left\| f''_{num} - f''_{exact} \right\|.$$

d) Repeat all the steps for the function $g(x) = 1/\cosh(x)$ on the interval $x \in [-\pi, \pi]$. What can you observe?

How do your results change if we consider g on $x \in [-4\pi, 4\pi]$ instead? Can you give an explanation?

Hint: $\cosh(x) = (e^x + e^{-x})/2, \quad \frac{d}{dx}\cosh(x) = \sinh(x), \quad \cosh(x)^2 = 1 - \sinh(x)^2.$

Discussion in the problem class Tuesday 3:45 pm/ Wednesday 2 pm , in room 3.069 in the Kollegiengebäude Mathematik 20.30.

Literature: L. Trefethen - Spectral Methods in Matlab (2000)