

Aspects of Numerical Time Integration — Exercise Sheet 06

June 18, 2015

We consider the nonlinear Schrödinger equation (NLS) in $(t, x) \in [0, T] \times \mathbb{K}$, i.e.

$$\text{for } t \in (0, T]: \partial_t u = i\Delta u + i|u|^2 u, \quad u(0, x) = u_0(x), \quad x \in \mathbb{K}. \quad (\text{NLS})$$

For practical implementation issues we introduce periodic boundary conditions for the solution of (NLS), i.e. we restrict ourselves to the torus $\mathbb{T}_L := [-L\pi, L\pi]$ for some $L > 0$ and we set $\mathbb{K} = \mathbb{T}_L$ in (NLS).

We choose L large enough, such that the boundary conditions are neglectable in the numerical solution.

Exercise 10:

As on exercise sheet 5 we choose the initial value $u_0(x) = \psi_\alpha(0, x)$, where

$$\psi_\alpha(t, x) = \frac{\sqrt{2\alpha}}{\cosh(x\sqrt{\alpha})} e^{i\alpha t}, \quad \alpha \in \mathbb{R}$$

is a solution of (NLS).

We set $T = 1, L = 2, \alpha = 8$.

In the following we will use spectral methods for the space discretization of (NLS) and a Strang splitting method for the time integration.

- a) On exercise sheet 5 we computed the error of the numerical solution using the approximate L^2 norm given by

$$\|f\|_{L^2} \approx \sqrt{h} \|\tilde{f}\|_{l^2},$$

where $f \in L^2$, h is the spatial step size corresponding to the spatial grid $x_j, j = 0, \dots, N-1$ and $\tilde{f} = (f(x_j))_{j=0}^{N-1}$.

How would you define the approximate H^1 norm for some $g \in H^1$ using spectral methods?

- b) Modify your MATLAB code of exercise 9, such that your code produces also an order plot for the temporal order of the numerical solution using the time step sizes $\tau_1 = 2^{-6}, \tau_2 = 2^{-8}, \tau_3 = 2^{-10}$ and $\tau_4 = 2^{-12}$, **additionally** to the order plot for the spatial order of the numerical solution with $N_1 = 32, N_2 = 64, N_3 = 128$ and $N_4 = 256$ grid points. You can use a loop running over the different time step sizes. Then fixing a particular $\tau_m, m = 1, 2, 3, 4$ you can run over the different $N_k, k = 1, 2, 3, 4$.

For the error computation use the approximate H^1 norm.

- c) Again as in exercise 9 create an efficiency plot for the efficiency of the Strang splitting method for each N (fix N , look at errors for various τ vs. elapsed time) and for each τ (fix τ , look at errors for various N vs. elapsed time).

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In the following we consider again problem (NLS) but now with true (not artificial) periodic boundary conditions. Then it is not that simple to find an analytic formula for the exact solution.

Therefore as on exercise sheet 2 in order to compute a reference solution we use a time integration method of higher order than the Strang splitting method.

Here we will use the splitting method of Yoshida, a 4th order time integration scheme which is given as follows:

Let $Au := i\Delta u$, $B(u)u := i|u|^2u$. Then $u_{n+1} \approx u(t_{n+1})$ is defined as

$$u_{n+1} = e^{c_4 A \tau} \circ e^{d_3 B \tau} \circ e^{c_3 A \tau} \circ e^{d_2 B \tau} \circ e^{c_2 A \tau} \circ e^{d_1 B \tau} \circ e^{c_1 A \tau} u_n, \quad (1)$$

where

$$\begin{aligned} c_1 = c_4 &= \frac{1}{2(2 - 2^{1/3})}, & c_2 = c_3 &= \frac{1 - 2^{1/3}}{2(2 - 2^{1/3})} \\ d_1 = d_3 &= \frac{1}{2 - 2^{1/3}}, & d_2 &= -\frac{2^{1/3}}{2 - 2^{1/3}}. \end{aligned}$$

Exercise 11:

We now set $\mathbb{K} = \mathbb{T} = [-\pi, \pi]$ and $T = 1$. Furthermore we choose the periodic initial value

$$u_0(x) = e^{2ix}(1 + 2 \cos(x)).$$

Repeat the steps of exercise 10 but now use the Yoshida splitting method as a reference method with time step size $\tau_m^{ref} := \tau_m/8, m = 1, 2, 3, 4$.

As before create order plots for the temporal and spatial order of the H^1 error of the numerical solution as well as the efficiency plots from before.