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Aspects of Numerical Time Integration — Exercise Sheet 07 July 14, 2015

We consider the nonlinear Schrödinger equation (NLS) in $(t, x) \in [0, T] \times \mathbb{T}^d$, i.e.

$$\text{for } t \in (0, T]: \quad \partial_t u = i\Delta u - i\alpha |u|^2 u, \quad u(0, x) = u_0(x), \quad x \in \mathbb{T}^d, \quad \alpha \in \mathbb{R}. \tag{NLS}$$

With the NLS the Hamiltonian (energy) H is associated. It is given by

$$H(u, \overline{u}) := \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} |\nabla u(x)|^2 + \frac{\alpha}{2} |u(x)|^4 dx. \tag{1}$$

<u>Exercise 12:</u>

a) Show that for the solution u of (NLS) the L^2 norm is conserved for all times $t \in [0, T]$, i.e. show that

$$\|u(t)\|_{L^2} = \|u(0)\|_{L^2}, \qquad \forall t \in [0, T].$$

b) Show that the Hamiltonian energy associated to the solution u of (NLS) is conserved for all times $t \in [0, T]$, i.e. show that

$$H\Big(u(t), \overline{u}(t)\Big) = H\Big(u(0), \overline{u}(0)\Big), \qquad \forall t \in [0, T].$$

Now we want to check in how far these conservation properties also hold numerically.

<u>Exercise 13:</u>

Set $d = 1$, $T = 1000$, $L = 0.11$, $\alpha = -1$ and let $\mathbb{T} := [-\pi/L, \pi/L]$.

Furthermore we choose the initial data

$$u_0(x) = \frac{\sqrt{(2)}}{\cosh(x)}, \quad x \in \mathbb{T}.$$

Implement the Strang splitting method applied to the NLS with the pseudospectral space discretization with $N = 256$ grid points.

Choose time step sizes $\tau_1 = 0.01$, $\tau_2 = 0.05$, $\tau_3 = 0.1$.

a) Plot the evolution of the L^2 norm error of the solution over all times $t_n = n\tau \in [0, T]$, $n = 0, 1, 2, \ldots$ for $\tau = \tau_j$, $j = 1, 2, 3$, i.e. plot

$$|\|u(t_n)\|_{L^2} - \|u(0)\|_{L^2}|, \quad \forall t_n \in [0, T].$$

b) Plot the evolution of the energy error of the solution over all times $t_n = n\tau \in [0, T]$, $n = 0, 1, 2, \ldots$ for $\tau = \tau_j$, $j = 1, 2, 3$, i.e. plot

$$|H(u(t_n), \overline{u}(t_n)) - H(u(0), \overline{u}(0))|, \quad \forall t_n \in [0, T].$$

Can you explain the behaviour of the energy for the different time step sizes?

please turn over →

Now we want to consider the NLS for $d = 2$. We set $\alpha = 0.12$.

Exercise 14:

Implement the Strang splitting method applied to the NLS for the with the pseudospectral space discretization with on $x = (x_1, x_2) \in \mathbb{T}^2 = [-\pi, \pi]^2$ with $N = 128$ grid points in the x_1- and x_2- direction.

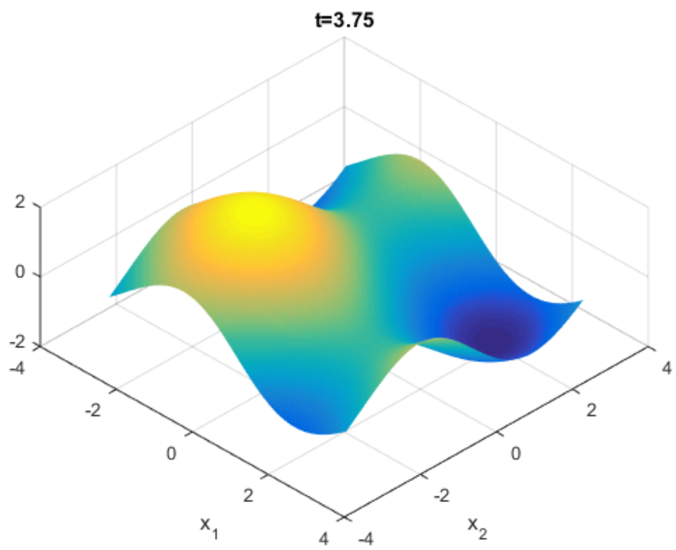
For the implementation use the `meshgrid` command in MATLAB .

Use the time step size $\tau = 0.01$.

We choose the initial value

$$u_0(x) = (\cos(x_1) + \cos(x_2)) + i(\sin(x_1) + \sin(x_2)).$$

a) Plot the numerical Strang splitting solution with time step size $\tau = 0.01$ using the `surf` command in MATLAB in the time interval $t \in [0, T]$ for $T = 10$.



b) Repeat the steps of exercise 13 for the time step sizes $\tau_1 = 0.01$, $\tau_2 = 0.05$, $\tau_3 = 0.1$, $\tau_4 = 0.25$, $\tau_5 = 0.4$, $\tau_6 = 0.5$ and plot for each time step size $\tau = \tau_j$, $j = 1, 2, 3, 4, 5, 6$ the evolution of the norm and energy error over all times $t \in [0, T]$ for $T = 100$.

Can you see the norm and energy conservation also in this 2D example?

What can you observe?

Discussion in the problem class Tuesday 3:45 pm/ Wednesday 2 pm , in room 3.069 in the Kollegiengebäude Mathematik 20.30.