Karlsruher Institut für Technologie

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Splitting Methods — Exercise Sheet 1 October 27, 2014

Exercise 1: (Fail of Euler Methods)

The explicit and implicit Euler method are simple methods to integrate ODEs numerically but there are a lot of examples where these methods fail.

(a) Consider the ODE

$$\dot{y}(t) = -3y(t), \quad y(0) = 1, \qquad t \in [0, 5]$$

and implement the explicit Euler method in MATLAB . Use time step sizes $h = 0.7$, $h = 0.5$, $h = 0.1$ and see how the numerical solution behaves. Compare the numerical solution with the exact solution $y(t) = e^{-3t}$.

(b) Consider the ODE

$$\dot{y}(t) = Ay(t), \quad y(0) = [1, 0, \ldots, 0]^T \in \mathbb{R}^N, \qquad t \in [0, 1]$$

with

$$A = \begin{pmatrix} & & & 1 \\ & & \iddots & \\ & 1 & & \\ -1 & 1 & \ldots & 1 \end{pmatrix}$$

and implement the implicit Euler method in MATLAB using time step size $h = 0.01$ and dimension $N = 10, 100, 1000, 10000$ of the system. What can you observe concerning the time MATLAB needs to compute the solution?

Exercise 2: (Representation of the exact solution of a linear ODE)

(a) Show that for $t > 0$ the exact solution of a differential equation

$$y'(t) = Ly(t), \quad y(0) = y_0, \qquad L \in \mathbb{R}^{n \times n}, n > 0$$

is given by

$$y(t) = \exp(Lt)y_0. \tag{1}$$

Hint: Consider the Taylor expansion of $y(t)$ and use that $y'(t) = f\big(y(t)\big) := Ly(t)$. How does $y''(t)$ look like? For the representation of $\frac{d^k}{dt^k}y(t),\ k > 1$ use an induction argument.

(b) Consider the differential equation

$$z''(t) = -\omega^2 z(t),\ t > 0, \quad z(0) = 1,\ z'(0) = 0, \qquad \omega \in \mathbb{R}, \tag{H}$$

which describes the movement of a *harmonic* oscillator. Rewrite equation (H) as a first order system of differential equations, i.e. set

$$y(t) := [z(t), z'(t)]^T$$

and find a matrix $L \in \mathbb{R}^{2\times 2}$ such that

$$y'(t) = Ly(t),\ t > 0, \qquad y(0) = y_0 := [z(0), z'(0)]^T. \tag{2}$$

(c) in MATLAB , plot the exact solution of (2) with $\omega = 1, 5, 10$ in the time interval $t \in [0, 10]$ by applying (1). Here please make use of

- a `for` loop
- the MATLAB built-in *Matrix exponential* function `expm`

Discussion in the problem class wednesday 3:45 pm, in room 1C-03 in building Allianzgebäude 5.20.