Karlsruher Institut für Technologie

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Splitting Methods — Exercise Sheet 11 January 30, 2015

On this exercise sheet we want to get more familiar with symplectic mappings.

For $y = (p, q) \in \mathbb{R}^{2d}$ we consider a Hamiltonian system

$$\dot{y} = J^{-1}\nabla H(y) =: X_H(y), \quad y(0) = (p_0, q_0) \in \mathbb{R}^{2d}, \qquad J = \begin{bmatrix} 0 & I_d \\ -I_d & 0 \end{bmatrix}. \tag{1}$$

Exercise 21: ♣ (implicit midpoint rule)

We consider a Hamiltonian $H : \mathbb{R}^{2d} \to \mathbb{R}$, $H \in C^2(\mathbb{R}^{2d}; \mathbb{R})$.

a) ♣ Show the identity $J^T = J^{-1} = -J$.

b) Show that the implicit midpoint rule defined by the mapping $y_{n+1} = \Phi^\tau_M(y_n)$ such that

$$y_{n+1} = y_n + \tau J^{-1}\nabla H\left(\frac{y_n + y_{n+1}}{2}\right)$$

is a symplectic mapping.

Hint: Use part a).

Exercise 22: (implicit midpoint rule applied to the harmonic oscillator)

As before on exercise sheet 8 we want to consider the Hamiltonian

$$H(p, q) = \frac{1}{2}(p^2 + q^2), \quad p, q \in \mathbb{R}$$

of a harmonic oscillator. We have seen that the Hamiltonian system in terms of p and q reads

$$\begin{aligned} \dot{p} &= -q, \\ \dot{q} &= +p, \end{aligned} \quad (p(0), q(0)) = (p_0, q_0).$$

a) Which approaches do you know for doing one step in an **implicit** method?

b) Formulate the implicit midpoint rule in terms of p_n and q_n for this Hamiltonian system. Can you find a representation of the method such that

$$(p_{n+1}, q_{n+1}) = \Psi(p_n, q_n),$$

where the right hand side only depends on p_n and q_n?

c) Implement this method in MATLAB with time step size τ on the time interval $t \in [0, T]$. We set $t_n = n\tau,\ n = 0, 1, 2, \ldots$.

Consider the initial data $(p_0, q_0) = (0, 4)$ and

- check the order of the numerical error (order plot!) of the numerical solution for $T = 1$ and suitable time step sizes $\tau < 1$. Note that by exercise 17 a) you can give the exact solution explicitly.
- check the conservation of the Hamiltonian up to time $T = 20$. (plot the difference $|H(p_n, q_n) - H(p_0, q_0)|$ over all times t_n.)
- Compare your results with the results of the symplectic Euler method and the Störmer-Verlet method. Which method gives the best energy conservation in this example? Which one has the best numerical error order?

Discussion in the problem class wednesday 3:45 pm, in room 1C-03 in building Allianzgebäude 5.20.

♣ : Please try to do exercises marked with ♣ at home.