

Splitting Methods — Exercise Sheet 2

October 30, 2014

Exercise 3:

Consider the differential equation

$$z''(t) = -\omega^2 z(t) - \alpha z'(t), \quad z(0) = 1, \quad z'(0) = 0, \quad \omega \in \mathbb{R}, \quad \alpha > 0, \quad (\text{D})$$

which describes the movement of a *damped* oscillator.

- a) Rewrite equation (D) as a first order system of differential equations, i.e. set $y(t) := [z(t), z'(t)]^T$ and find a matrix $L \in \mathbb{R}^{2 \times 2}$ such that

$$y'(t) = Ly(t), \quad y(0) = y_0 := [z(0), z'(0)]^T. \quad (1)$$

- b) How does the explicit Euler scheme look like applied to problem (1)?
c) Do the matrices

$$\tilde{A} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} 0 & 0 \\ 0 & -\alpha \end{pmatrix}$$

commute, i.e. is $\tilde{A}\tilde{B} = \tilde{B}\tilde{A}$?

- d) Find a decomposition of L , i.e. matrices A and B with $L = A + B$, such that A and B commute.

Hint: Rewrite $L = (L - X) + X$ with a general matrix $X \in \mathbb{R}^{2 \times 2}$ and set $A := L - X$, $B := X$. Then compare the entries of AB with the entries of BA and solve a linear system of equations for the coefficients of X .

Exercise 4: (First Lie Splitting)

Now we want to get used to the application of a Lie Splitting Method to an ODE. Therefore consider the ODE (1) on the time interval $t \in [0, 16]$ and set $\omega = 5$, $\alpha = 1$.

- a) In MATLAB implement the explicit Euler method for the system (1) with time step size $h = 2^{-6}$. Plot the numerical solution `y_euler` and the exact solution `y_exact` (use matrix exponential `expm`) into the same figure. In another figure plot the error of `y_euler`.
b) Now implement the Lie splitting method for the system (1) with time step size $h = 2^{-6}$. First consider a splitting $L = \tilde{A} + \tilde{B}$ and plot the numerical solution `y_Lie` and its error into the same figures as in a) in order to compare the Lie splitting method with the explicit Euler method.
c) How does the error of the Lie splitting method change if we use a decomposition $L = A + B$ with the matrices A and B from exercise 3 d) instead.

Exercise 5: (Local error of explicit Euler method and Lie Splitting method)

Consider the ODE

$$\dot{y} = f(y) = f_1(y) + f_2(y), \quad y(0) = y^0, \quad f, f_1, f_2 \text{ "smooth"}$$

with flow $\varphi_f^h : y_0 \mapsto y(h)$. We say a numerical method $\Phi^h : y^n \mapsto y^{n+1}$ is consistent of order p if the local error satisfies

$$\left\| \Phi^h \left(y(t_n) \right) - \varphi_f^h \left(y(t_n) \right) \right\| \leq Ch^{p+1}.$$

- (a) Show that the explicit Euler method is consistent of order $p = 1$.
(b) Let $f_1(y) := Ay$, $f_2(y) = By$, $A, B \in \mathbb{R}^{n \times n}$.

Show that the Strang splitting method $\Phi_S^h(y_0) := \varphi_A^{h/2} \circ \varphi_B^h \circ \varphi_A^{h/2}(y_0)$ is consistent of order $p = 2$.

Hint: Taylor expansion of $y(h) = \varphi_L^h(y^0) = e^{Lh}y^0$, $L \in \mathbb{R}^{n \times n}$.

Discussion in the problem class wednesday 3:45 pm, in room 1C-03 in building Allianzgebäude 5.20.