Splitting Methods — Exercise Sheet 4

November 14, 2014

We consider an ODE

\[ \dot{y} = Ay + By, \quad y(0) = y^0 \]  

with matrices \( A, B \in \mathbb{R}^{n \times n} \) and the corresponding subproblems

\[ \dot{w} = Aw, \quad w(0) = w^0 \]
\[ \dot{z} = Bz, \quad z(0) = z^0 \]

**Exercise 8:**  (Order of Convergence of the Strang splitting method)

In exercise 5 on exercise sheet 2 we have shown that the local error of the Strang splitting method

\[ \Phi_h^k(y^0) = e^{Ah/2}e^{Bh}e^{Ah/2}y^0 \]

satisfies

\[ \left\| \Phi_h^k(\cdot) - \Phi_h^k(y^0) \right\| = \left\| e^{(A+B)h}y^0 - e^{Ah/2}e^{Bh}e^{Ah/2}y^0 \right\| \leq Ch^3 \left( \| [B, [B, A]] \| + \| [A, [A, B]] \| \right), \]

where \( [A, B] := AB - BA \) is called the commutator of \( A \) and \( B \).

Furthermore we assume stability, i.e. there exist constants \( M_A, M_B, M_L \in \mathbb{R} \) such that

\[ \left\| e^{tA} \right\| \leq e^{M_A t}, \quad \left\| e^{tB} \right\| \leq e^{M_B t}, \quad \left\| e^{(A+B) t} \right\| \leq e^{M_L t}, \quad \forall t \geq 0. \]  

(2)

a) Formulate a theorem on the order of convergence, i.e. on the global error of the Strang splitting method.

b) Prove the theorem of a).

**Hint:** Proceed as in the proof of the theorem for the order of convergence of the Lie splitting method.

c) Why don’t we compute an “approximation” to \( \Phi_h^k(\cdot) \) just by \( \Phi_h^k(y^0) \)?
Now we want to prove some auxiliary results which we need to prove the Baker-Campbell-Hausdorff (BCH) formula. We define for $H, \Omega \in \mathbb{R}^{n \times n}$

$$\left( \frac{d}{d\Omega} \Omega^k \right)_H := \lim_{h \to 0} \frac{(\Omega + hH)^k - \Omega^k}{h} = H\Omega^{k-1} + \Omega H\Omega^{k-2} + \cdots + \Omega^{k-1}H$$

$$ad^0_\Omega(H) := id(H) = H, \quad ad^{i+1}_\Omega(H) = [\Omega, ad^i_\Omega(H)]$$

**Exercise 9:** ♣ (Proof of Lemma 3.3)
Show that

$$\left( \frac{d}{d\Omega} \Omega^k \right)_H = \sum_{i=0}^{k-1} \binom{k}{i+1} (ad^i_\Omega(H)) \Omega^{k-i-1} \quad \left( \begin{array}{c} l \\ m \end{array} \right) = \frac{l!}{(l-m)!m!}, \quad l, m \in \mathbb{N}, l \geq m.$$  

**Hint:** Induction, show that $\left( \frac{d}{d\Omega} \Omega \right)_H = \Omega \left( \frac{d}{d\Omega} \Omega^k \right)_H + \left( \frac{d}{d\Omega} \Omega \right)_H \Omega^k$.

**Exercise 10:** (Proof of Lemma 3.6)
Let by lemma 3.4

$$d \exp_\Omega(H) := \sum_{k=0}^{\infty} \frac{1}{(k+1)!} ad^k_\Omega(H). \quad (3)$$

Show that

a) if the eigenvalues of $ad_\Omega \neq 2l\pi i$, $l \in \mathbb{Z} \setminus \{0\}$, then $d \exp_\Omega$ is invertible.

**Hint:** Compute the eigenvalues of $d \exp_\Omega$.

b) ♣ for $\|\Omega\| < \pi$ its inverse is given by

$$d \exp^{-1}_\Omega(H) = \sum_{k=0}^{\infty} \frac{B_k}{k!} ad^k_\Omega(H),$$

where the Bernoulli numbers $B_k$ are defined by

$$\sum_{k=0}^{\infty} \frac{B_k}{k!} x^k = \frac{x}{e^x - 1}.$$

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**Discussion in the problem class**

Wednesday 3:45 pm, in room 1C-03 in building Allianzgebäude 5.20.

♣: Please try to do exercises marked with ♣ at home.