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Splitting Methods — Exercise Sheet 4 November 14, 2014

We consider an ODE

$$\dot{y} = Ay + By, \quad y(0) = y^0 \tag{1}$$

with matrices $A, B \in \mathbb{R}^{n\times n}$ and the corresponding subproblems

$$\dot{w} = Aw, \quad w(0) = w^0$$
$$\dot{z} = Bz, \quad z(0) = z^0$$

Exercise 8: (Order of Convergence of the Strang splitting method)

In exercise 5 on exercise sheet 2 we have shown that the *local* error of the Strang splitting method

$$\Phi_S^h(y^0) = e^{Ah/2}e^{Bh}e^{Ah/2}y^0$$

satisfies

$$\left\|\varphi^h_{A+B}\left(y^0\right) - \Phi_S^h\left(y^0\right))\right\| = \left\|e^{(A+B)h}y^0 - e^{Ah/2}e^{Bh}e^{Ah/2}y^0\right\| \leq Ch^3\left\|y^0\right\|\left(\|[B,[B,A]]\| + \|[A,[A,B]]\|\right),$$

where $[A, B] := AB - BA$ is called the *commutator* of A and B.

Furthermore we assume *stability*, i.e.there exist constants $M_A, M_B, M_L \in \mathbb{R}$ such that

$$\left\|e^{tA}\right\| \leq e^{tM_A}, \qquad \left\|e^{tB}\right\| \leq e^{tM_B}, \qquad \left\|e^{t(A+B)}\right\| \leq e^{tM_L}, \qquad \forall t \geq 0. \tag{2}$$

a) Formulate a theorem on the order of convergence, i.e. on the *global* error of the Strang splitting method.

b) Prove the theorem of a).
Hint: Proceed as in the proof of the theorem for the order of convergence of the Lie splitting method.

c) Why don't we compute an "approximation"to $\varphi^{t_n}_{A+B}(y^0)$ just by $\Phi_S^{t_n}(y^0)$?

please turn over →

Now we want to prove some auxiliary results which we need to prove the *Baker-Campbell-Hausdorff* (BCH) formula. We define for $H, \Omega \in \mathbb{R}^{n\times n}$

$$\left(\frac{d}{d\Omega}\Omega^k\right)_H := \lim_{h\to 0}\frac{(\Omega+hH)^k-\Omega^k}{h} = H\Omega^{k-1}+\Omega H\Omega^{k-2}+\cdots+\Omega^{k-1}H$$
$$ad^0_\Omega(H) := id(H) = H, \qquad ad^{i+1}_\Omega(H) = [\Omega, ad^i_\Omega(H)]$$

<u>Exercise 9:</u> ♣ (Proof of Lemma 3.3)

Show that

$$\left(\frac{d}{d\Omega}\Omega^k\right)_H = \sum_{i=0}^{k-1}\binom{k}{i+1}\left(ad^i_\Omega(H)\right)\Omega^{k-i-1}, \qquad \binom{l}{m} = \frac{l!}{(l-m)!m!},\ l,m\in\mathbb{N}, l\geq m.$$

Hint: Induction, show that $\left(\frac{d}{d\Omega}\Omega\cdot\Omega^k\right)_H = \Omega\left(\frac{d}{d\Omega}\Omega^k\right)_H + \left(\frac{d}{d\Omega}\Omega\right)_H\Omega^k$.

<u>Exercise 10:</u> (Proof of Lemma 3.6)

Let by lemma 3.4

$$d\exp_\Omega(H) := \sum_{k=0}^{\infty}\frac{1}{(k+1)!}ad^k_\Omega(H). \tag{3}$$

Show that

a) if the eigenvalues of $ad_\Omega \neq 2l\pi i,\ l\in\mathbb{Z}\setminus\{0\}$, then $d\exp_\Omega$ is invertible.
   Hint: Compute the eigenvalues of $d\exp_\Omega$.

b) ♣ for $\|\Omega\| < \pi$ its inverse is given by

$$d\exp^{-1}_\Omega(H) = \sum_{k=0}^{\infty}\frac{B_k}{k!}ad^k_\Omega(H),$$

where the *Bernoulli numbers* B_k are defined by

$$\sum_{k=0}^{\infty}\frac{B_k}{k!}x^k = \frac{x}{e^x-1}.$$

Discussion in the problem class wednesday 3:45 pm, in room 1C-03 in building Allianzgebäude 5.20.

♣ : Please try to do exercises marked with ♣ at home.