Karlsruher Institut für Technologie

Department of Mathematics
Institute for Applied and
Numerical Mathematics

JProf. Dr. Katharina Schratz
Dipl-Math. techn. Patrick Krämer

Splitting Methods — Exercise Sheet 5 November 21, 2014

Let A, B two matrices. Then by Lemma 3.7

$$e^{tA}e^{tB} = e^{C(t)},$$

where $C(t)$ solves the initial value problem

$$\begin{aligned} \dot{C}(t) &= A + B + \frac{1}{2}[A - B, C(t)] + \sum_{k=2}^{\infty} \frac{B_k}{k!} ad^k_{C(t)}(A + B), \\ C(0) &= 0, \end{aligned} \tag{BCH}$$

where $B_2 = \frac{1}{6}$, $B_3 = 0$. We make the ansatz

$$C(t) = C_0 + tC_1 + t^2C_2 + t^3C_3 + t^4C_4 + \mathcal{O}\left(t^5\right)$$

which yields

$$\dot{C}(t) = C_1 + 2tC_2 + 3t^2C_3 + 4t^3C_4 + \mathcal{O}\left(t^5\right).$$

and by the initial value $C(0) = 0$ we have $C_0 = 0$.

Exercise 11: (BCH formula)

In the lecture we already determined the coefficients $C_1 = A + B$ and $C_2 = \frac{1}{2}[A, B]$.

a) Let X, Y, Z be matrices. Show the *Jacobi identity*

$$\Big[X, [Y, Z]\Big] + \Big[Z, [X, Y]\Big] + \Big[Y, [Z, X]\Big] = 0.$$

b) ♣ Determine the coefficients C_3 and C_4.

Hint: Use the Jacobi identity of part a) for a simplified representation of C_4.

c) Assume that $[A, B] = 0$. What can you then say about the convergence of the Lie splitting method applied to

$$\dot{y} = (A + B)y, \quad y(0) = y_0?$$

please turn over →

Now we want to introduce the concept of the *Lie derivative*.
We consider the model problem

$$\begin{aligned} \dot{y} &= f^{[1]}(y) + f^{[2]}(y), \quad f^{[i]} : \mathbb{R}^n \to \mathbb{R}^n \\ y(0) &= y_0 \in \mathbb{R}^n \end{aligned} \tag{1}$$

with flow $\varphi_t(y_0)$ and the corresponding subproblems

$$\dot{y} = f^{[1]}(y), \qquad \dot{y} = f^{[2]}(y)$$

with flows $\varphi_t^{[1]}$ and $\varphi_t^{[2]}$ respectively.
By the *BCH* formula we can derive order conditions for splitting methods applied to (1).
For this purpose we need the so called *Lie derivative*.

Definition 1 (Lie derivative)**.** *The Lie derivative D_g associated to $g : \mathbb{R}^n \to \mathbb{R}^n$ is defined by*

$$D_g := \sum_{j=1}^{n} g_j(y) \frac{\partial}{\partial y_j} = (g_1(y), g_2(y), \dots, g_n(y)) \begin{pmatrix} \frac{\partial}{\partial y_1} \\ \frac{\partial}{\partial y_2} \\ \vdots \\ \frac{\partial}{\partial y_n}. \end{pmatrix}$$

Applied to $F \in C^1(\mathbb{R}^n : \mathbb{R}^m)$ we have

$$D_g F(y) = F'(y) \cdot g(y) = \begin{bmatrix} \frac{\partial F_1}{\partial y_1}(y) & \cdots & \frac{\partial F_1}{\partial y_n}(y) \\ \vdots & & \vdots \\ \frac{\partial F_m}{\partial y_1}(y) & \cdots & \frac{\partial F_m}{\partial y_n}(y) \end{bmatrix} \begin{bmatrix} g_1(y) \\ \vdots \\ g_n(y) \end{bmatrix}$$

Remark: If we set $g = f^{[i]},\ i = 1, 2$ we write D_i instead of D_g.

Exercise 12: (Lie Derivative)

(a) Let $g(y_1, y_2) := \begin{pmatrix} y_1^2 \\ e^{y_2} \end{pmatrix}$ and $F(y_1, y_2) := \begin{pmatrix} \sin(y_1) + \cos(y_2) \\ 3y_1^2 + y_2 - 3 \\ e^{5y_1} + y_1 y_2^3 \end{pmatrix}$.

Compute the Lie derivative $D_g F(y_1, y_2)$ associated to g in F explicitly.

Let $F \in C^\infty(\mathbb{R}^n : \mathbb{R}^m)$ arbitrary and refer to model problem (1).

(b) Show that

$$\frac{d}{dt} F\left(\varphi_t^{[i]}(y_0)\right) = (D_i F)\left(\varphi_t^{[i]}(y_0)\right), \quad i = 1, 2.$$

(c) ♣ Prove that for $k \geq 0$ the identity

$$\frac{d^k}{dt^k} F\left(\varphi_t^{[i]}(y_0)\right) = (D_i^k F)\left(\varphi_t^{[i]}(y_0)\right), \quad i = 1, 2$$

holds by induction.

Hint: Chain rule and product rule of differentiation.

Discussion in the problem class wednesday 3:45 pm, in room 1C-03 in building Allianzgebäude 5.20.

♣ : Please try to do exercises marked with ♣ at home.