Karlsruher Institut für Technologie

Department of Mathematics
Institute for Applied and
Numerical Mathematics

JProf. Dr. Katharina Schratz
Dipl-Math. techn. Patrick Krämer

Splitting Methods — Exercise Sheet 6 November 28, 2014

Consider the differential equation

$$\dot{y} = f^{[1]}(y) + f^{[2]}(y), \quad y(0) = y_0.$$

We set $f^{[1]}(y) = y^2,\ f^{[2]}(y) = 1$.

Exercise 13: (Lemma by Gröbner)

a) Verify that the flows $\varphi^{[j]}$ corresponding to $f^{[j]},\ j = 1,2$ are given by

$$\varphi_s^{[1]}(y_0) = \frac{y_0}{1 - sy_0}, \qquad \varphi_t^{[2]}(y_0) = t + y_0.$$

b) Check that the Lemma by Gröbner applies to this example, i.e. show that

$$\left(\varphi_s^{[1]} \circ \varphi_t^{[2]}\right)(y_0) = \exp(tD_2)\exp(sD_1)Id(y_0),$$

where D_j is the Lie derivative corresponding to $f^{[j]}$.
Hint: In the lecture we showed the representation

$$\varphi_t^{[j]}(y) = \exp(tD_j)Id(y).$$

c) Show by induction that in general for flows $\varphi_{s_j}^{[j]},\ j = 1, \ldots, m$ there holds

$$\left(\varphi_{s_m}^{[m]} \circ \cdots \circ \varphi_{s_2}^{[2]} \circ \varphi_{s_1}^{[1]}\right)(y_0) = \exp(s_1 D_1)\exp(s_2 D_2)\cdots\exp(s_m D_m)Id(y_0)$$

Hint: Use the identity for a sufficiently smooth F

$$F\left(\varphi_t^{[i]}(y_0)\right) = \exp(tD_i)F(y_0).$$

Exercise 14:

Let $f^{[1]}(y)$ and $f^{[2]}(y)$ be defined on an open set.

With the help of the BCH formula, show that the corresponding flows $\varphi_s^{[1]}$ and $\varphi_t^{[2]}$ commute if and only if $[D_1, D_2] = 0$.

Discussion in the problem class wednesday 3:45 pm, in room 1C-03 in building Allianzgebäude 5.20.

♣ : Please try to do exercises marked with ♣ at home.