Karlsruher Institut für Technologie

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Splitting Methods — Exercise Sheet 8 January 4, 2015

On this exercise sheet we want to get familiar with Hamiltonian systems of the form

$$\dot{y}(t) = J^{-1}\nabla H(y(t)), \quad y(0) = y_0, \quad J = \begin{bmatrix} 0 & I_d \\ -I_d & 0 \end{bmatrix},$$

where $y = (p, q) \in \mathbb{R}^{2d}$, $H \colon \mathbb{R}^{2d} \mapsto \mathbb{R}$ a smooth Hamiltonian, I_d the d-dimensional identity matrix and $\nabla = (\partial_{p_1}, \dots, \partial_{p_d}, \partial_{q_1}, \dots, \partial_{q_d})$.

Then this system reads

$$\begin{aligned} \dot{p}_i &= -\frac{\partial H}{\partial q_i}(p,q) \\ \dot{q}_i &= +\frac{\partial H}{\partial p_i}(p,q) \end{aligned}, \quad i = 1, \dots, d. \tag{1}$$

In a Hamiltonian system we can identify q with the *position* and p with the *momentum* (or setting the mass $m \equiv 1$ with the *velocity* of a particle).

Another characteristic point of a hamiltonian system is the preservation of energy, i.e. for the Hamiltonian we have $H(p(t), q(t)) = H(p(0), q(0))$ for all times t.

Exercise 17: (explicit and symplectic Euler method)

The Hamiltonian of a 1D harmonic oscillator ($d = 1$) is given by

$$H(p,q) = \frac{1}{2}(p^2 + q^2), \; p, q \in \mathbb{R}.$$

(a) Write down the corresponding Hamiltonian system in the form of (1), such that

$$\begin{aligned} \dot{p} &= f(q) \\ \dot{q} &= g(p). \end{aligned}$$

Can you give an expression for the *exact* solution of this system?

Now we want to test two numerical methods on this system and check the preservation of the Hamiltonian for both integrators. We approximate $p_n \approx p(t_n)$ and $q_n \approx q(t_n)$, $t_n = nh, n = 0, 1, 2, \dots$.

Consider $(p_0, q_0) = (0, 4)$, $h = 0.1$ and the time interval $t \in [0, 20]$.

(b) In MATLAB implement the explicit Euler method for this system.

(c) In the same MATLAB file also implement the *symplectic* Euler method (also called *semi-implicit* Euler method) as follows:

$$\begin{aligned} p_{n+1} &= p_n + hf(q_n) \\ q_{n+1} &= q_n + hg(p_{n+1}) \end{aligned}, \quad p_n, q_n \text{ given.}$$

(d) Compare the numerical solutions of both integrators with the exact solution and plot the error $r_n = \|(p_n, q_n) - (p(t_n), q(t_n))\|$ over all times t_n in a semilogarithmic plot. What can you see?

(e) For both methods check the conservation of energy, i.e. for both methods plot $|H(p_n, q_n) - H(p_0, q_0)|$ over all times t_n in a semilogarithmic plot. What can you see?

(f) Which numerical order do you expect for the symplectic Euler method? Create an order plot for the symplectic Euler method and check this order numerically.

Discussion in the problem class wednesday 3:45 pm, in room 1C-03 in building Allianzgebäude 5.20.

♣ : Please try to do exercises marked with ♣ at home.