

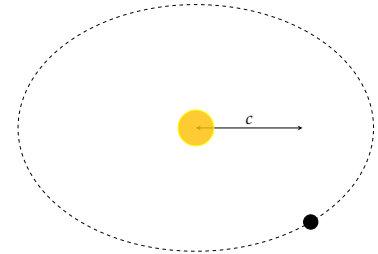
Splitting Methods — Exercise Sheet 9

January 9, 2015

On this exercise sheet we want to simulate the 2-body Kepler problem which describes the motion of two bodies attracting each other. A simple example is the motion of a planet on an orbit around the sun.

We fix one body as the center of our coordinate system. Then the motion of the second body will stay in a 2D plane.

In the following we set $q = (q_1, q_2)$ the position of the second body and $p = (p_1, p_2)$ its momentum. Newton's laws of motion then yield the differential equations



$$\ddot{q}_i = -\frac{q_i}{(q_1^2 + q_2^2)^{3/2}}, \quad q_i(0), \dot{q}_i(0) \text{ given}, \quad i = 1, 2.$$

Exercise 18: (conservation of Hamiltonian and angular momentum)

a) Show that this is equivalent to a Hamiltonian system with

$$H(p, q) = \frac{1}{2}(p_1^2 + p_2^2) - \frac{1}{\sqrt{q_1^2 + q_2^2}} =: T(p) + P(q).$$

b) Show that the Hamiltonian H and also the angular momentum $L(p, q) = q_1 p_2 - q_2 p_1$ are conserved over all times t , i.e. show that

$$H(p(t), q(t)) = H(p(0), q(0)), \quad L(p(t), q(t)) = L(p(0), q(0)) \quad \forall t \geq 0.$$

Exercise 19: ♣ (simulation of the 2-body Kepler problem)

Let $c = 0.6$ be the eccentricity of the orbit, $q(0) = (1 - c, 0)$, $p(0) = (0, \sqrt{\frac{1+c}{1-c}})$. Let $h = 0.01$ and $t_n = nh$, $n = 0, 1, 2, 3, \dots$ and let $\tilde{p}_n \approx p(t_n)$, $\tilde{q}_n \approx q(t_n)$.

a) In MATLAB implement the explicit and the symplectic Euler method applied to the Hamiltonian system of Ex. 18 a) for the time interval $t \in [0, 10\pi]$ and simulate the motion of the two bodies numerically.

b) Plot the total energy $H(\tilde{p}_n, \tilde{q}_n)$ and the energy error $|H(\tilde{p}_n, \tilde{q}_n) - H(\tilde{p}_0, \tilde{q}_0)|$ of the numerical solutions of both methods for all n .

Do the same for the angular momentum $L(\tilde{p}_n, \tilde{q}_n)$ and the error of the angular momentum $|L(\tilde{p}_n, \tilde{q}_n) - L(\tilde{p}_0, \tilde{q}_0)|$. What can you see?

c) Another popular method for the numerical solution of a Hamiltonian system

$$\dot{p} = -\frac{\partial P}{\partial q}(q), \quad \dot{q} = +\frac{\partial T}{\partial p}(p)$$

is the *Störmer-Verlet* method defined as follows: Let p_0, q_0 be given. Then

$$p_{n+\frac{1}{2}} = p_n - \frac{h}{2} \frac{\partial P}{\partial q}(q_n)$$

$$q_{n+1} = q_n + h \frac{\partial T}{\partial p}(p_{n+\frac{1}{2}})$$

$$p_{n+1} = p_{n+\frac{1}{2}} - \frac{h}{2} \frac{\partial P}{\partial q}(q_{n+1}).$$

Also implement the Störmer-Verlet method and compare the findings of parts a) and b) with the numerical solution obtained by the Störmer-Verlet method. What can you see?

d) Repeat all the computation with step size $h = 0.05$. How do your results change?

Discussion in the problem class wednesday 3:45 pm, in room 1C-03 in building Allianzgebäude 5.20.