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Aspects of numerical time integration — Exercise Sheet 4 May 2, 2014

On this exercise sheet we want to learn how to visualize numerical results on functions $f(t,x)$ which depend on time t and space x. On previous exercise sheets we have already plotted the numerical solution of an ordinary differential equation of the form $y' = Ay$ and obtained a static plot, which described the evolution $y(t)$ of the solution over the time t.

Now we want to create animated plots where we can see how $f(t,x)$ evolves over time.

Furthermore we want to apply spectral methods and finite difference methods to solve the transport equation

$$\partial_t u(t,x) = -\partial_x u(t,x),\ u(0) = u_0(x)$$

on the Torus $\mathbb{T} \subset \mathbb{R}$.

Exercise 8:

Consider the function $f(t,x) = \sin(t) \cdot \cos(x-t)$ for $x \in [-2\pi, 2\pi]$ and $t \in [0, 10]$.

a) create a vector `tvec` containing all time steps of the interval $[0, 10]$ with time step size $\tau = 0.1$ and discretize the spatial interval $[-\pi, \pi]$ with $N = 32$ grid points and spatiel step size $h = \frac{\pi-(-\pi)}{N} = \frac{2\pi}{N}$.

b) use a `for` loop to plot $f($`tvec`(j),`xvec`$)$ at each time step `tvec`(j).

c) compute $\partial_x f$ and $\partial_{xx} f$ with finite differences and spectral methods using the MATLAB functions from exercise sheet 3.

d) in the same `for` loop as in b) also plot the numerical spatial derivatives of f. Do you see what you expected to see?

e) modify the MATLAB functions `dx_sp(funvec)` and `dxx_sp(funvec)` from exercise sheet 3 such that the numerical differentiation also works for functions $g(x)$ which are periodic on a space interval $[a,b]$ of arbitrary length $b-a$ and not only for intervals of length $b-a = 2\pi$.
(HINT: Think about how the spectral differentiation method was motivated and how its scheme on exercise sheet 3 is related to the spatial interval $[a,b]$. Keyword: Transformation of the interval.)

Exercise 9: (the linear transport equation)

Consider the linear transport equation

$$\partial_t u(t,x) = -c\partial_x u(t,x), \quad u(0,x) = g(x), \quad x \in \mathbb{T},\ t \in [0,T]. \tag{1}$$

a) what is the characteristic property of its analytical solution $u(t,x)$ concerning the initial value $u(0,x)$?

Let us denote $A := -c\partial_x$ the linear differentiation operator w.r.t. x. Then we can rewrite (1) as

$$\partial_t u(t,x) = Au(t,x), \quad u(0,x) = g(x).$$

Formally its exact solution is given by

$$u(t,x) = e^{At} g(x).$$

Consider a space discretization $x_j = jh,\ j = 0, \ldots, N$ for some $N \in \mathbb{N}$ and denote $u_j(t) = u(t, x_j)$. Assume $u_0(t) = u_N(t)$ for all t and set $\tilde{u}(t) = \left(u_j(t)\right)_{j=0}^{N-1}$.

Then in Fourier space we can also solve this equation exactly w.r.t. time t. Given some time step size τ the numerical solution after one step can be computed by

$$\mathcal{F}_N \tilde{u}(\tau) = e^{-c\cdot(iK)\tau} * \mathcal{F}_N \tilde{u}(0), \quad K = \left[0 : \frac{N}{2} - 1, 0, -\frac{N}{2} + 1 : -1\right]. \tag{2}$$

Again $*$ denotes the pointwise product of two vectors and e^K has to be understood likewise. Therefore we have

$$\tilde{u}(\tau) = \mathcal{F}_N^{-1} \left(e^{-c\cdot(iK)\tau} * \mathcal{F}_N \tilde{u}(0)\right), \quad K = \left[0 : \frac{N}{2} - 1, 0, -\frac{N}{2} + 1 : -1\right].$$

(b) write a MATLAB function to solve (1) and make use of the spectral method above.

(c) compute the numerical solution of the transport equation for an arbitrary $c > 0$ of your choice with initial value $g(x) = e^{-x^2/2}$ on the spatial interval $x \in [-6\pi, 6\pi]$ with $N = 1024$ grid points in the time interval $t \in [0, 10]$ with time step size $\tau = 10^{-3}$ and plot the result in an animated plot.

(d) in the plot, add $g(x - ct)$.

(e) repeat the computations for the initial value $g(x) = e^{\sin(x)}$.

Discussion in the exercises tuesday 3:45 pm, the 06.05.2014 in room K2 (Kronenstraße 32).