On this exercise sheet we want to learn how to visualize numerical results on functions \( f(t, x) \) which depend on time \( t \) and space \( x \). On previous exercise sheets we have already plotted the numerical solution of an ordinary differential equation of the form \( y' = Ay \) and obtained a static plot, which described the evolution \( y(t) \) of the solution over the time \( t \).

Now we want to create animated plots where we can see how \( f(t, x) \) evolves over time.

Furthermore we want to apply spectral methods and finite difference methods to solve the transport equation
\[
\partial_t u(t, x) = -\partial_x u(t, x), \quad u(0) = u_0(x)
\]
on the Torus \( T \subset \mathbb{R} \).

**Exercise 8:**

Consider the function \( f(t, x) = \sin(t) \cdot \cos(x - t) \) for \( x \in [-2\pi, 2\pi] \) and \( t \in [0, 10] \).

a) create a vector \( tvec \) containing all time steps of the interval \([0,10]\) with time step size \( \tau = 0.1 \) and discretize the spatial interval \([-\pi, \pi]\) with \( N = 32 \) grid points and spatial step size \( h = \frac{\pi - (-\pi)}{N} = \frac{2\pi}{N} \).

b) use a \( \text{for} \) loop to plot \( f(tvec(j), xvec) \) at each time step \( tvec(j) \).

c) compute \( \partial_x f \) and \( \partial_{xx} f \) with finite differences and spectral methods using the MATLAB functions from exercise sheet 3.

d) in the same \( \text{for} \) loop as in b) also plot the numerical spatial derivatives of \( f \). Do you see what you expected to see?

e) modify the MATLAB functions \( \text{dx_sp}(funvec) \) and \( \text{dxx_sp}(funvec) \) from exercise sheet 3 such that the numerical differentiation also works for functions \( g(x) \) which are periodic on a space interval \([a, b]\) of arbitrary length \( b-a \) and not only for intervals of length \( b-a = 2\pi \).

(HINT: Think about how the spectral differentiation method was motivated and how its scheme on exercise sheet 3 is related to the spatial interval \([a, b]\). Keyword: Transformation of the interval.)

**Exercise 9:** (the linear transport equation)

Consider the linear transport equation
\[
\partial_t u(t, x) = -c \partial_x u(t, x), \quad u(0, x) = g(x), \quad x \in T, \; t \in [0, T]. \tag{1}
\]

a) what is the characteristic property of its analytical solution \( u(t, x) \) concerning the initial value \( u(0, x) \)?

Let us denote \( A := -c \partial_x \) the linear differentiation operator w.r.t. \( x \). Then we can rewrite (1) as
\[
\partial_t u(t, x) = Au(t, x), \quad u(0, x) = g(x).
\]

Formally its exact solution is given by
\[
u(t, x) = e^{At}g(x).
\]

Consider a space discretization \( x_j = jh, \; j = 0, \ldots, N \) for some \( N \in \mathbb{N} \) and denote \( u_j(t) = u(t, x_j) \). Assume \( u_0(t) = u_{N}(t) \) for all \( t \) and set \( \tilde{u}(t) = (u_j(t))_{j=0}^{N-1} \).

Then in Fourier space we can also solve this equation exactly w.r.t. time \( t \). Given some time step size \( \tau \) the numerical solution after one step can be computed by
\[
\tilde{F}_N \tilde{u}(\tau) = e^{-c(iK)\tau} \ast \tilde{F}_N \tilde{u}(0), \quad K = \left[ 0 : \frac{N}{2} - 1, 0, -\frac{N}{2} + 1 : -1 \right]. \tag{2}
\]

Again \( \ast \) denotes the pointwise product of two vectors and \( e^{ik} \) has to be understood likewise. Therefore we have
\[
\tilde{u}(\tau) = \tilde{F}_N^{-1} \left( e^{-c(iK)\tau} \ast \tilde{F}_N \tilde{u}(0) \right), \quad K = \left[ 0 : \frac{N}{2} - 1, 0, -\frac{N}{2} + 1 : -1 \right].
\]
(b) write a MATLAB function to solve (1) and make use of the spectral method above.

(c) compute the numerical solution of the transport equation for an arbitrary $c > 0$ of your choice with initial value $g(x) = e^{-x^2/2}$ on the spatial interval $x \in [-6\pi, 6\pi]$ with $N = 1024$ grid points in the time interval $t \in [0, 10]$ with time step size $\tau = 10^{-3}$ and plot the result in an animated plot.

(d) in the plot, add $g(x - ct)$.

(e) repeat the computations for the initial value $g(x) = e^{\sin(x)}$.

Discussion in the exercises tuesday 3:45 pm, the 06.05.2014 in room K2 (Kronenstraße 32).