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Aspects of numerical time integration — Exercise Sheet 8 June 24, 2014

On this exercise sheet we want to analyze the numerical order of the Lie and Strang splitting method applied to the nonlinear Schrödinger equation (NLS)

$$\partial_t u = i\Delta u + i|u|^2 u, \qquad u(0,x) = g(x), \qquad x \in \mathbb{T}, \quad t \in [0,T], \tag{1}$$

on the torus $\mathbb{T} := [-10\pi, 10\pi]$. For implementation reasons we assume periodic boundary conditions.

Discretize space with $N = 2048$ grid points and choose a time step size of $\tau_0 = 2^{-12}$ such that $t_n = n\tau,\ n = 0,1,2,\ldots$.

In order to obtain a reference solution to compare the numerical solution from above with, we consider the 4th order splitting method by Yoshida (c.f. [1]) which is given by

$$u_{n+1} = e^{c_4 A\tau} \circ e^{d_3 B\tau} \circ e^{c_3 A\tau} \circ e^{d_2 B\tau} \circ e^{c_2 A\tau} \circ e^{d_1 B\tau} \circ e^{c_1 A\tau} u_n, \tag{2}$$

where

$$c_1 = c_4 = \frac{1}{2\left(2-2^{1/3}\right)}, \qquad c_2 = c_3 = \frac{1-2^{1/3}}{2\left(2-2^{1/3}\right)}$$

$$d_1 = d_3 = \frac{1}{2-2^{1/3}}, \qquad d_2 = -\frac{2^{1/3}}{2-2^{1/3}}.$$

Here we set

$$Au := i\Delta u \quad \text{and} \quad B(u)u := i|u|^2 u.$$

Exercise 14: (Reference solution and order plot)

We consider the initial value

$$g(x) = \frac{\sqrt{2}}{\cosh(x)}$$

and set $\tau = \tau_0 := 2^{-12}$.

a) Write a MATLAB function which applies the Yoshida splitting method (2) to the NLS (1) in order to obtain a numerical reference solution.

b) Create an order plot for the Lie and Strang splitting method applied to the NLS (1). Therefore set $T = 5$ and consider the time step sizes $\tau_j = 2^j \cdot \tau_0,\ j = 1,\ldots,M$. Proceed as in exercise 4 on exercise sheet 2:

   i) compute numerical solutions y_{Lie} and y_{Strang} of the NLS (1) applying the Lie and Strang splitting method on the time interval $[0,T]$ with time step size τ_0.

   ii) compute a reference solution y_{Ref} of the NLS (1) applying the Yoshida splitting method (2) on the time interval $[0,T]$ with a finer time step size $\tilde{\tau}_0 := \frac{\tau_0}{2}$.

   iii) compute the maximal H^1 error of the Lie and Strang splitting solution, i.e. for the Lie splitting

   $$\text{err}_0 = \max_{t\in[0,T]} \|y_{\text{Lie}}(t) - y_{\text{Ref}}(t)\|_{H^1}.$$

   iv) save the error in an array and repeat steps i)-iii) for the step sizes $\tau_j,\ j = 1,\ldots,M$.

   v) plot the array $[\tau_0,\ldots,\tau_M]$ and the corresponding errors $[\text{err}_0,\ldots,\text{err}_M]$ of the Lie and the Strang splitting respectively in a loglog-Plot

c) Which order of the error do you expect for the Lie and the Strang splitting method? Add lines into the plot with slopes which represent the expected order.

Discussion in the exercises tuesday, the 24.06.2014 at 3:45 pm in room K2 (Kronenstraße 32).

[1] Haruo Yoshida, Construction of higher order symplectic integrators, National Astronomical Observatory, Mitaka, Tokyo 181, Japan, 1990