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Aspects of numerical time integration — Exercise Sheet 9 July 1, 2014

On this exercise sheet we want to analyze the smoothness of the Lie and Strang splitting solution of the nonlinear Schrödinger equation (NLS)

$$\partial_t u = i\Delta u + i|u|^2 u, \qquad u(0,x) = g(x), \qquad x \in \mathbb{T}, \quad t \in [0,T],$$

on the torus $\mathbb{T}^d \subset \mathbb{R}^d$ and how it depends on the regularity of the initial value $u(0,x)$.

As a little extra task we furthermore want to implement a solver for the NLS in $\mathbb{T}^2$.

For implementation reasons we assume periodic boundary conditions.

We know that for $s > \frac{d}{2}$ given there holds

$$\|u_{exact}(t) - u_{Lie}(t)\|_{H^s} \le C\tau \|u(0)\|_{H^{s+2}}, \quad \forall t \in [0,T]$$

and

$$\|u_{exact}(t) - u_{Strang}(t)\|_{H^s} \le C\tau^2 \|u(0)\|_{H^{s+4}}, \quad \forall t \in [0,T].$$

The question now is: How does the order of the error change if we use in case of Lie splitting an initial value $u(0) := \Psi_L \in H^{s+1}(\mathbb{T}) \setminus H^{s+2}(\mathbb{T})$, i.e. is there any $p > 0$ which satifies

$$\|u_{exact}(t) - u_{Lie}(t)\|_{H^s} \le C\tau^p \|\Psi_L\|_{H^{s+1}}, \quad \forall t \in [0,T] \quad ?$$

For the Strang splitting the question is the same for initial values $u(0) := \Psi_S \in H^{s+3}(\mathbb{T}) \setminus H^{s+4}(\mathbb{T})$

Discretize space with $N = 2048$ grid points and choose a time step size of $\tau = 2^{-12}$ such that $t_n = n\tau,\ n = 0, 1, 2, \ldots$.

In order to obtain a reference solution, use Yoshida's splitting method (see exercise sheet 8).

Exercise 15: (Regularity of the solution)

Let $\mathbb{T} := [-10\pi, 10\pi]$ and $s = 1$, so we are interested in the H^1 error of the Lie and Strang splitting solutions.

a) Try to find/construct functions $\Psi_L \in H^{1+1}(\mathbb{T}) \setminus H^{1+2}(\mathbb{T})$ and $\Psi_S \in H^{1+3}(\mathbb{T}) \setminus H^{1+4}(\mathbb{T})$.

b) Apply the Lie and Strang splitting method to the NLS with initial values Ψ_L for the Lie and Ψ_S for the Strang splitting method and create an order plot for both methods. Compute the error by

$$\frac{\|u_{ref}(t) - u_{num}(t)\|_{H^1}}{\|u(0)\|_{H^{1+k}}},$$

where u_{num} is the numerical solution obtained by Lie or Strang splitting and u_{ref} is the reference solution obtained by Yoshida's splitting method.

Which order of the error can you see? Depending on the choice of your initial value multiple solutions are possible.

c) What happens if we use an initial value which is only in $L^2(\mathbb{T}) \setminus H^1(\mathbb{T})$?

(please turn over)

Exercise 16: (NLS in 2D)

In this exercise we want to write a MATLAB function in order to solve the NLS

$$\partial_t u = i\Delta u - i|u|^2 u, \qquad u(0,x) = g(x), \qquad x \in \mathbb{T}, \quad t \in [0,T],$$

in 2D. We are on the torus $\mathbb{T}^2 := [-16\pi, 16\pi] \times [-16\pi, 16\pi]$ and again assume periodic boundary conditions. We discretize space with $N = 1024$ grid points in x and y direction.

a) Create the spatial grid with the MATLAB `meshgrid(xvector,yvector)` command and plot the function

$$g(x,y) = 2e^{-\frac{x^2+y^2}{4}}$$

using the `surf(xdata,ydata,zdata)` command.

b) What is the representation of the 2D laplacian in Fourier space? Use again the `meshgrid` command to create the grid for the Fourier numbers in x and y direction.

c) Write a MATLAB function to solve the NLS in $\mathbb{T}^2$ and apply a Strang splitting method using spectral methods. To compute the Fourier tranform and its inverse use the MATLAB commands `fft2()` and `ifft2()`.

d) Solve the NLS on the time interval $[0, 10]$ with initial value $u(0,x,y) = g(x,y)$ and time step size $\tau = 0.01$ and plot your results using `surf`.

Discussion in the exercises tuesday, the 01.07.2014 at 3:45 pm in room K2 (Kronenstraße 32).