An abstract framework for inverse wave problems with applications

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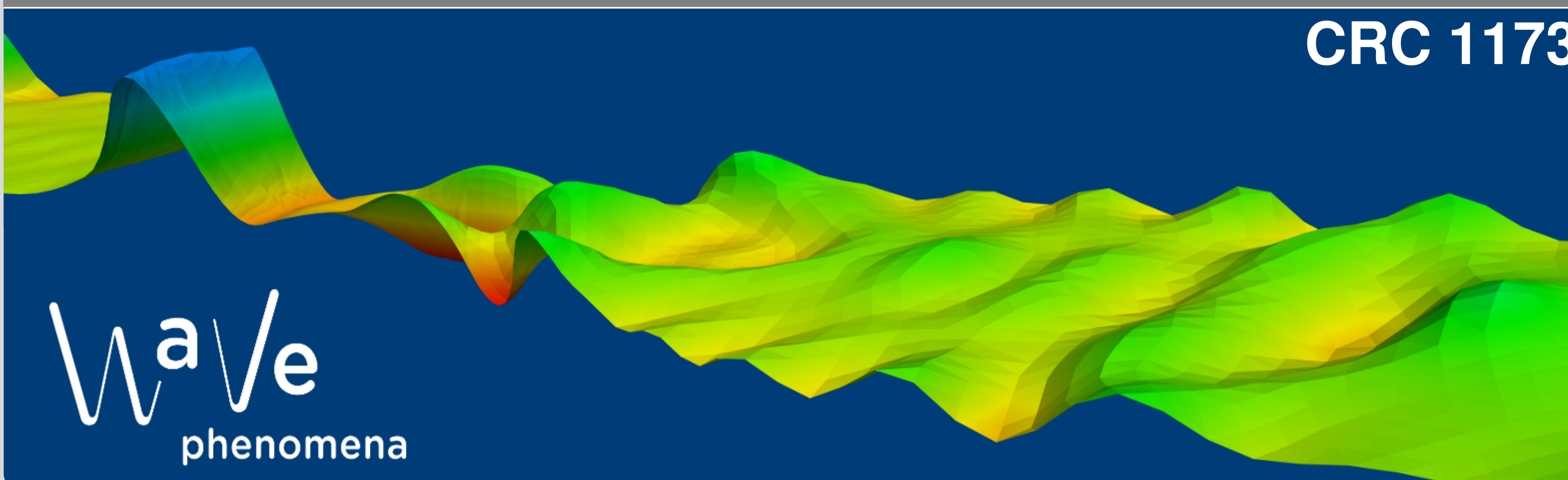

Outline

Direct Problem

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Summary

Direct Problem

Abstract evolution equations

Consider

$$Bu'(t) + Au(t) = f(t), \quad t > 0, \quad u(0) = u_0,$$

where

- $A \colon \mathcal{D}(A) \subset X \to X$
  is an (unbounded) operator on the Hilbert space X,
- $B \in \mathcal{L}(X)$ is positive self-adjoint, and
- $f \colon [0, \infty[\to X, \; u_0 \in X$.

Note: Neither A nor B depend on time t.

Blazek, Stolk & Symes: *A mathematical framework for inverse wave problems in heterogeneous media*, Inverse Problems 29 (2013).

Example: the Maxwell system

$\mathbf{E} = \mathbf{E}(t, \mathbf{x})$ and $\mathbf{H} = \mathbf{H}(t, \mathbf{x})$ electric and magnetic fields, resp.

$$\underbrace{\begin{pmatrix} \varepsilon I & 0 \\ 0 & \mu I \end{pmatrix}}_{= B} \begin{pmatrix} \mathbf{E}' \\ \mathbf{H}' \end{pmatrix} + \underbrace{\begin{pmatrix} \sigma I & -\mathrm{curl}_x \\ \mathrm{curl}_x & 0 \end{pmatrix}}_{= A} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \underbrace{\begin{pmatrix} -\mathbf{J}_e \\ \mathbf{J}_m \end{pmatrix}}_{= f} \quad \text{in }]0, T[\times D$$

with bc: $\mathbf{n} \times \mathbf{E} = 0$ on $]0, T[\times \partial D$

ic: $\mathbf{E}(0, \cdot) = \mathbf{e}_0$, $\mathbf{H}(0, \cdot) = \mathbf{h}_0$

where $\mathbf{J}_{e/m} = \mathbf{J}_{e/m}(t, x)$ current/magnetic density

$\varepsilon = \varepsilon(x)$ permittivity

$\mu = \mu(x)$ permeability

$\sigma = \sigma(x)$ conductivity

$D \subset \mathbb{R}^3$ bounded Lipschitz domain (or an exterior of such a domain).

Example (continued)

- $X = L^2(D, \mathbb{R}^3) \times L^2(D, \mathbb{R}^3)$
- $\mathcal{D}(A) = H_0(\mathrm{curl}, D) \times H(\mathrm{curl}, D)$
- If $0 < \mathrm{c} \leq \varepsilon, \mu$ a.e. then $B \in \mathcal{L}(X)$ is positive self-adjoint.

Note: The elastic wave equation can also be formulated in this abstract framework.

Classical solution: existence, uniqueness, and stability

$$Bu'(t) + Au(t) = f(t), \quad t > 0, \quad u(0) = u_0,$$

Theorem Let $u_0 \in \mathcal{D}(A)$ and $f \in W^{1,1}_{\text{loc}}([0,\infty[, X)$. Further, let A be maximal monotone, that is,

$$\langle Av, v\rangle_X \geq 0 \ \forall v \in \mathcal{D}(A) \quad \text{and} \quad A + I : \mathcal{D}(A) \to X \ \text{is onto.}$$

Then, the evolution problem has a unique classical solution $u \in \mathcal{C}([0,\infty[, \mathcal{D}(A)) \cap \mathcal{C}^1([0,\infty[, X)$ satisfying

$$\|u'(t)\|_X + \|u(t)\|_{\mathcal{D}(A)} \lesssim \|u_0\|_{\mathcal{D}(A)} + \|f\|_{W^{1,1}([0,t],X)} \quad \text{for all } t > 0.$$

Proof: 1. $B^{-1}A$ is maximal monotone. 2. Apply the Hille-Yosida theorem to

$$u'(t) + B^{-1}Au(t) = B^{-1}f(t), \quad t > 0, \quad u(0) = u_0.$$

Remark: $B^{-1}A$ generates a $\mathcal{C}^0$ semigroup of contractions in $(X, \langle\cdot,\cdot\rangle_B)$.

Weak solution: existence, uniqueness, and stability

Corollary Let $u_0 \in X$, $f \in L^1_{\mathrm{loc}}([0,\infty[, X)$, A and B as above.
Then, the evolution problem has a unique weak or mild solution
$u \in \mathcal{C}([0,\infty[, X)$: $u(0) = u_0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle Bu(t), v\rangle_X + \langle u(t), A^* v\rangle_X = \langle f(t), v\rangle_X \quad \text{for a.a. } t > 0, \quad \forall v \in \mathcal{D}(A^*).$$

Further,

$$\|u(t)\|_X \lesssim \|u_0\|_X + \|f\|_{L^1([0,t],X)} \quad \text{for all } t \geq 0.$$

Remark: A maximal monotone

$\Longrightarrow \mathcal{D}(A)$ dense, A closed, and A^* maximal monotone as well

Example: the Maxwell system (revisited)

$$\underbrace{\begin{pmatrix} \varepsilon I & 0 \\ 0 & \mu I \end{pmatrix}}_{=B} \begin{pmatrix} \mathbf{E}' \\ \mathbf{H}' \end{pmatrix} + \underbrace{\begin{pmatrix} \sigma I & -\mathrm{curl}_x \\ \mathrm{curl}_x & 0 \end{pmatrix}}_{=A} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \underbrace{\begin{pmatrix} -\mathbf{J}_e \\ \mathbf{J}_m \end{pmatrix}}_{=f}$$

- $X = L^2(D, \mathbb{R}^3) \times L^2(D, \mathbb{R}^3)$
- $\mathcal{D}(A) = H_0(\mathrm{curl}, D) \times H(\mathrm{curl}, D)$

Lemma If $0 \leq \sigma \in L^\infty(D)$ then A is maximal monotone.

Proof: For $(\mathcal{E}, \mathcal{H})^\top \in \mathcal{D}(A)$ we have

$$\left\langle A \begin{pmatrix} \mathcal{E} \\ \mathcal{H} \end{pmatrix}, \begin{pmatrix} \mathcal{E} \\ \mathcal{H} \end{pmatrix} \right\rangle_X = \int_D \left[(\sigma \mathcal{E} - \mathrm{curl}_x \mathcal{H}) \cdot \mathcal{E} + \mathrm{curl}_x \mathcal{E} \cdot \mathcal{H} \right] \mathrm{d}x = \int_D \sigma \, |\mathcal{E}|^2 \mathrm{d}x \geq 0$$

by Green's theorem (no boundary term appears as $\mathcal{E} \in H_0(\mathrm{curl}, D)$).

Inverse Problem

Parameter-to-solution map

For $u_0 \in \mathcal{D}(A)$, $f \in W^{1,1}_{\mathrm{loc}}([0,\infty[, X)$, A maximal monotone, consider

$$F \colon \mathcal{D}(F) \subset \mathcal{L}(X) \to \mathcal{C}([0,T], X), \quad B \mapsto u$$

where $T > 0$,

$$\mathcal{D}(F) \subset \mathcal{B} = \big\{ B \in \mathcal{L}(X) : B = B^*,\ \beta \|v\|_X^2 \leq \langle Bv, v\rangle_X \ \forall v \in X \big\}, \quad \beta > 0,$$

and u solves

$$Bu'(t) + Au(t) = f(t), \quad t \in \,]0,T[, \quad u(0) = u_0.$$

Local ill-posedness of parameter identification

Theorem The inverse problem

$$F(\cdot) = u$$

is locally ill-posed at any $\widehat{B} \in \mathcal{D}(F)$ satisfying $F(\widehat{B}) = u$ if there are an $\widehat{r} > 0$ and a family $\{E_k\} \subset \mathcal{L}(X)$ of positive semi-definite operators s.t.

$$\forall r \in]0, \widehat{r}]\colon \widehat{B} + rE_k \in \mathcal{D}(F), \quad \|E_k\| \sim 1, \quad E_k \xrightarrow{k\to\infty} 0 \text{ pointwise.}$$

Proof: Set $B_k := \widehat{B} + rE_k$. Since

$$\|B_k - \widehat{B}\| \sim r$$

it remains to show that

$$F(B_k) \xrightarrow{k\to\infty} F(\widehat{B}) \text{ in } \mathcal{C}([0,T], X).$$

Fréchet differentiability

Theorem $F\colon \mathcal{B} \subset \mathcal{L}(X) \to \mathcal{C}([0,T],X)$, $B \mapsto u$, is Fréchet differentiable at any $B \in \operatorname{int}(\mathcal{B})$ with derivative

$$F'(B)H = u_H|_{[0,T]}$$

where u_H is the weak solution of

$$Bu_H'(t) + Au_H(t) = -Hu'(t), \quad t > 0, \quad u_H(0) = 0,$$

with $u = F(B)$, i.e., classical solution of $Bu'(t) + Au(t) = f(t)$, $u(0) = u_0$.

Remark: Fréchet differentiability of

$$F\colon \mathcal{B} \subset \mathcal{L}(X) \to \mathcal{C}^1([0,T],X) \cap \mathcal{C}([0,T],\mathcal{D}(A))$$

requires stronger regularity assumptions on f and u_0, namely

$$f \in W^{3,1}([0,T],X),\ v := B^{-1}\big(Au_0 - f(0)\big) \in \mathcal{D}(A),\ B^{-1}\big(Av - f'(0)\big) \in \mathcal{D}(A).$$

Adjoint of the Fréchet derivative

Here, $F'(B)\colon \mathcal{L}(X) \to L^2([0,T],X)$ linear and bounded. Then,

$$F'(B)^*\colon L^2([0,T],X) \to \mathcal{L}(X)^*.$$

Theorem We have that

$$[F'(B)^* g]H = \int_0^T \langle Hu'(t), w(t)\rangle_X \,\mathrm{d}t$$

where w is the weak solution of the backwards equation

$$Bw'(t) - A^* w(t) = g(t), \quad t \in\,]0,T[, \quad w(T) = 0,$$

and $u = F(B)$, i.e., classical solution of $Bu'(t) + Au(t) = f(t)$, $u(0) = u_0$.

Remark: Set $\widetilde{w}(t) = w(T-t)$. Then, the backwards eqn. is equivalent to

$$B\widetilde{w}'(t) + A^*\widetilde{w}(t) = -g(T-t), \quad t \in\,]0,T[, \quad \widetilde{w}(0) = 0.$$

Inverse Electromagnetic Scattering

Parameter-to-solution map

$$\begin{pmatrix} \varepsilon I & 0 \\ 0 & \mu I \end{pmatrix} \begin{pmatrix} \mathbf{E}' \\ \mathbf{H}' \end{pmatrix} + \begin{pmatrix} \sigma I & -\mathrm{curl}_x \\ \mathrm{curl}_x & 0 \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} -\mathbf{J}_e \\ \mathbf{J}_m \end{pmatrix}, \quad \begin{pmatrix} \mathbf{E}(0) \\ \mathbf{H}(0) \end{pmatrix} = \begin{pmatrix} \mathbf{e}_0 \\ \mathbf{h}_0 \end{pmatrix}$$

- $X = L^2(D, \mathbb{R}^3) \times L^2(D, \mathbb{R}^3)$
- $\mathcal{D}(A) = H_0(\mathrm{curl}, D) \times H(\mathrm{curl}, D)$
- $(\mathbf{e}_0, \mathbf{h}_0)^\top \in \mathcal{D}(A)$ and $(\mathbf{J}_e, \mathbf{J}_m)^\top \in W^{1,1}\big([0,T], X\big)$ for $T > 0$
- $\mathcal{P} := \big\{(\varepsilon, \mu)^\top \in L^\infty(D)^2 : 0 < \mathrm{c} \le \varepsilon, \mu \text{ a.e.}\big\}$

Then, the parameter-to-solution map is well defined:

$$\Phi\colon \mathcal{P} \subset L^\infty(D)^2 \to \mathcal{C}\big([0,T], X\big), \quad (\varepsilon, \mu)^\top \mapsto (\mathbf{E}, \mathbf{H})^\top.$$

Factorization of the parameter-to-solution map

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We factorize

$$\Phi = F \circ V$$

where

$$F\colon \mathcal{B} \subset \mathcal{L}(X) \to \mathcal{C}\big([0,T], X\big), \quad B \mapsto (\mathbf{E}, \mathbf{H})^\top,$$

and

$$V\colon \mathcal{P} \subset L^\infty(D)^2 \to \mathcal{B}, \quad (\varepsilon, \mu)^\top \mapsto \begin{pmatrix} \varepsilon I & 0 \\ 0 & \mu I \end{pmatrix}.$$

Note that F is the mapping considered in the abstract theory.

Inverse electromagnetic scattering is locally ill-posed

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Theorem The inverse problem

$$\Phi(\varepsilon, \mu) = (\mathbf{E}, \mathbf{H})^\top$$

is locally ill-posed at any interior point of $\mathcal{P}$.

Proof: Let

$$e_k := \chi_{\mathrm{B}_{1/k}(\xi)} \quad \text{for one fixed } \xi \in D$$

and define

$$E_k := \begin{pmatrix} e_k I & 0 \\ 0 & e_k I \end{pmatrix} = V(e_k, e_k).$$

Then, $\|E_k\| = 1$, $E_k \xrightarrow{k\to\infty} 0$ pointwise, and, for any $r > 0$,

$$\Phi(\varepsilon + re_k, \mu + re_k) = F\big(V(\varepsilon,\mu) + rE_k\big) \xrightarrow{k\to\infty} F\big(V(\varepsilon,\mu)\big) = \Phi(\varepsilon,\mu)$$

but

$$\varepsilon + re_k \not\to \varepsilon \quad \text{as well as} \quad \mu + re_k \not\to \mu \quad \text{(due to } \|e_k\|_\infty = 1\text{).}$$

Fréchet derivative of the electromagnetic forward map

Theorem The parameter-to-solution map

$$\Phi\colon \mathcal{P} \subset L^\infty(D)^2 \to \mathcal{C}\big([0,T],X\big)$$

is Fréchet differentiable at $(\varepsilon,\mu)^\top$. In fact,

$$\Phi'(\varepsilon,\mu)\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \begin{pmatrix} \overline{\mathbf{E}} \\ \overline{\mathbf{H}} \end{pmatrix}$$

where $(\overline{\mathbf{E}},\overline{\mathbf{H}})^\top \in \mathcal{C}\big([0,T],X\big)$ is the weak solution of

$$\begin{pmatrix} \varepsilon I & 0 \\ 0 & \mu I \end{pmatrix}\begin{pmatrix} \overline{\mathbf{E}}' \\ \overline{\mathbf{H}}' \end{pmatrix} + \begin{pmatrix} \sigma I & -\mathrm{curl}_x \\ \mathrm{curl}_x & 0 \end{pmatrix}\begin{pmatrix} \overline{\mathbf{E}} \\ \overline{\mathbf{H}} \end{pmatrix} = \begin{pmatrix} -h_1 \mathbf{E}' \\ -h_2 \mathbf{H}' \end{pmatrix},$$

in $[0,T]\times D$ with $\overline{\mathbf{E}}(0) = \overline{\mathbf{H}}(0) = \mathbf{0}$. Here, $(\mathbf{E},\mathbf{H})^\top = \Phi(\varepsilon,\mu)$.

Proof: $\Phi'(\varepsilon,\mu)\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = F'\big(V(\varepsilon,\mu)\big)V'(\varepsilon,\mu)\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = F'\big(V(\varepsilon,\mu)\big)V(h_1,h_2)$

Adjoint of the Fréchet derivative

Theorem The dual operator $\Phi'(\varepsilon,\mu)^*\colon L^2([0,T],X) \to \big(L^\infty(D)^*\big)^2$ is given by

$$\Phi'(\varepsilon,\mu)^* \begin{pmatrix} \mathbf{g}_E \\ \mathbf{g}_H \end{pmatrix} = \begin{pmatrix} \int_0^T \mathbf{E}'(t,\cdot)\cdot \overline{\mathbf{E}}(t,\cdot)\mathrm{d}t \\ \int_0^T \mathbf{H}'(t,\cdot)\cdot \overline{\mathbf{H}}(t,\cdot)\mathrm{d}t \end{pmatrix} \in L^1(D)^2$$

where $(\overline{\mathbf{E}},\overline{\mathbf{H}})^\top \in \mathcal{C}([0,T],X)$ uniquely solves

$$\begin{pmatrix} \varepsilon I & 0 \\ 0 & \mu I \end{pmatrix} \begin{pmatrix} \overline{\mathbf{E}}' \\ \overline{\mathbf{H}}' \end{pmatrix} + \begin{pmatrix} -\sigma I & -\mathrm{curl}_x \\ \mathrm{curl}_x & 0 \end{pmatrix} \begin{pmatrix} \overline{\mathbf{E}} \\ \overline{\mathbf{H}} \end{pmatrix} = \begin{pmatrix} \mathbf{g}_E \\ \mathbf{g}_H \end{pmatrix}$$

with $(\overline{\mathbf{E}}(T),\overline{\mathbf{H}}(T))^\top = (0,0)^\top$.

Summary

Things to remember

$$Bu'(t) + Au(t) = f(t), \;\; t > 0, \quad u(0) = u_0$$

$B \in \mathcal{L}(X)$ pos. self-adjoint; $A\colon \mathcal{D}(A) \subset X \to X$ maximal monotone; X Hilbert space

We have investigated $F\colon B \mapsto u$ and shown the

- Fréchet-differentiability of F,
- Local ill-posedness of $F(\cdot) = u$,

and applied our abstract findings to

- Maxwell's system (inverse electromagnetic scattering).

Final note: The abstract theory applies to the elastic wave equation (seismic tomography) as well.

A. Kirsch, A. Rieder: *Inverse problems for abstract evolution equations with applications in electrodynamics and elasticity*, Inverse Problems (2016), to appear