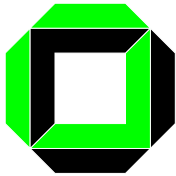


An Overview on Mathematical Methods in Tomography

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Preface

Focus: 2D X-ray computerized tomography

CT variants not addressed here:

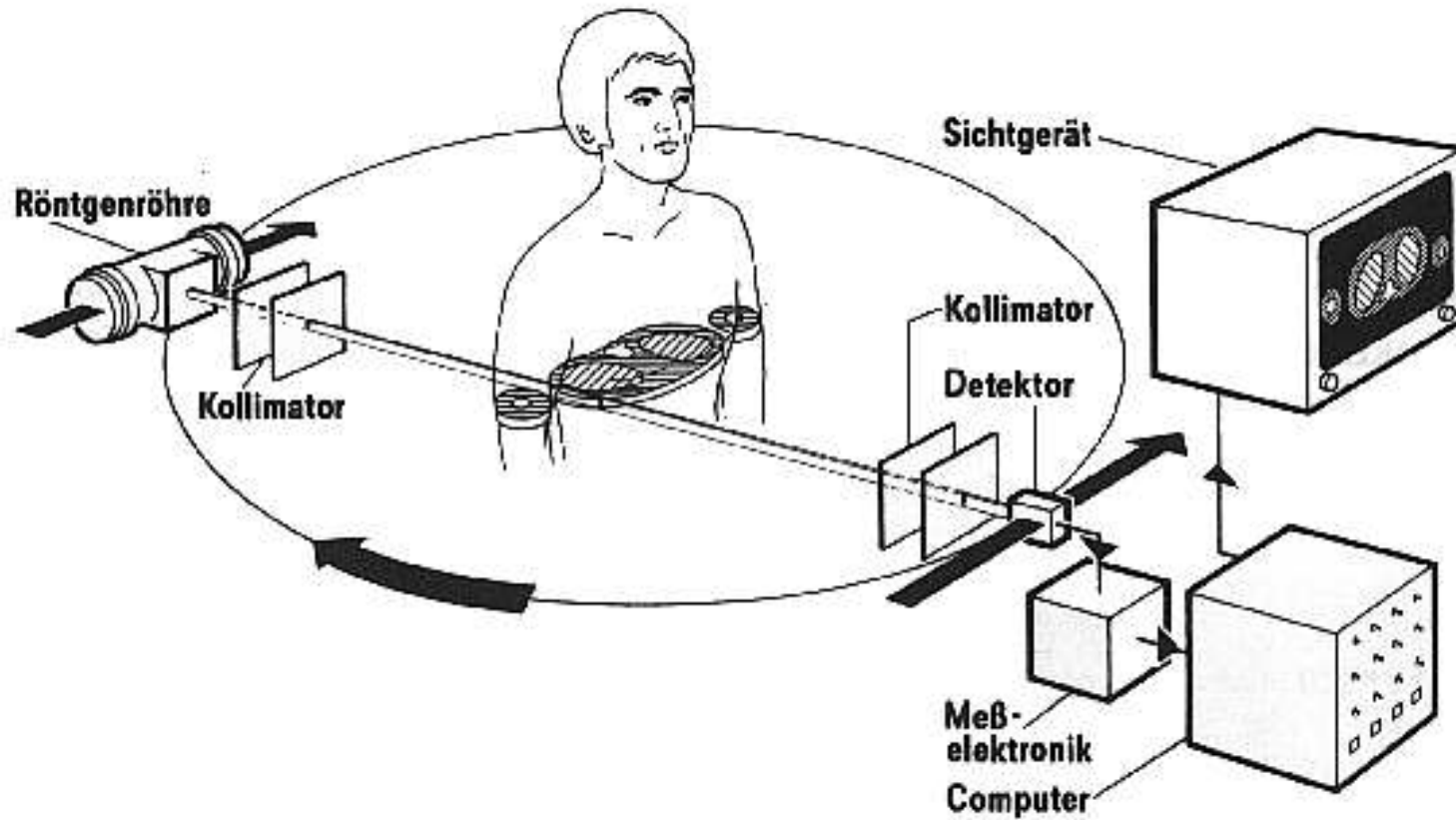
- 3D CT
- SPECT
- Doppler CT
- Diffusive (optical) CT
- MR imaging
- Impedance CT
- Ultrasound CT

F. Natterer, F. Wübbeling: *Mathematical Methods in Image Reconstruction*,
SIAM 2001

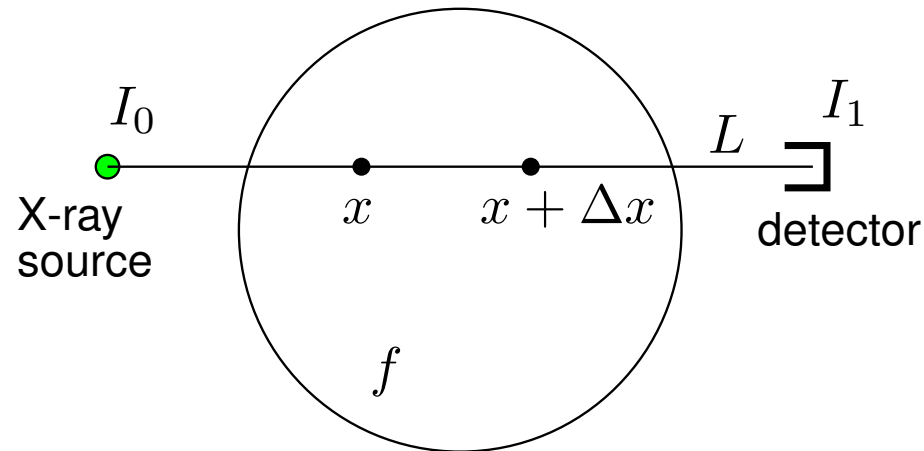
Contents

- Mathematical model for CT: the Radon transform
- Inversion formula: global and local tomography
- Non-uniqueness for discrete data
- Approximate inversion
- Filtered backprojection algorithm
- Computational examples

Principle of CT scanning device



The mathematical model



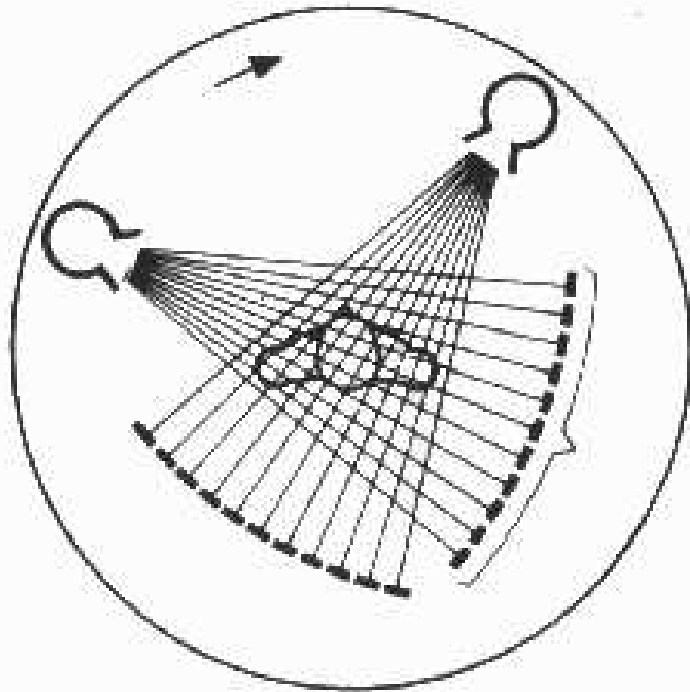
physical assumpt.: $I(x + \Delta x) - I(x) = -f(x) \|\Delta x\| I(x)$

$$\frac{I(x + \Delta x) - I(x)}{\|\Delta x\|} = -f(x) I(x)$$

$$\Delta x \rightarrow 0 \implies \partial_L \ln I(x) = -f(x)$$

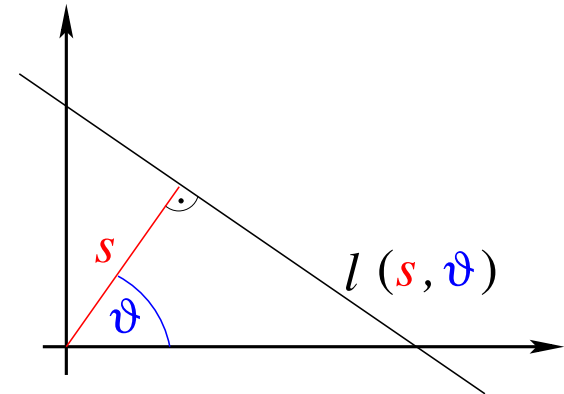
$$\int_L f(x) d\sigma(x) = \ln(I_0/I_1)$$

CT scanning geometries



2D-Radon-Transform (parallel scanning geometry)

$$\mathbf{R}f(s, \vartheta) := \int_{l(s, \vartheta) \cap \Omega} f(x) d\sigma(x)$$



tomographic inversion: $\mathbf{R}f(s, \vartheta) = g(s, \vartheta)$

$$\mathbf{R} : L^2(\Omega) \rightarrow L^2(Z), \quad Z = [-1, 1] \times [0, \pi]$$

Johann Radon 1917, A. M. Cormack 1963, G. N. Hounsfield 1967

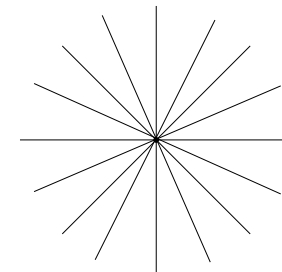
Inversion formula

Riesz potential $\Lambda^\alpha : H^t(\mathbb{R}^d) \rightarrow H^{t-\alpha}(\mathbb{R}^d)$, $\alpha > -d$

$$\widehat{\Lambda^\alpha f}(\xi) := \|\xi\|^\alpha \widehat{f}(\xi), \quad \Lambda^\alpha = (-\Delta)^{\alpha/2}$$

backprojection $\mathbf{R}^* : L^2(Z) \rightarrow L^2(\Omega)$

$$\mathbf{R}^* g(x) = \int_0^\pi g(x^t \omega(\vartheta), \vartheta) d\vartheta$$



$$\Lambda^\alpha f = \frac{1}{2\pi} \mathbf{R}^* \Lambda_s^{1+\alpha} \mathbf{R} f, \quad f \in L^2(\Omega)$$

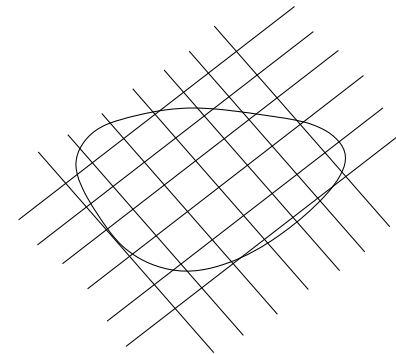
$\alpha = 0$: Radon 1917, general result: Smith, Solomon and Wagner 1977

Global and local tomography

$$\Lambda^\alpha f = \frac{1}{2\pi} \mathbf{R}^* \Lambda_s^{1+\alpha} \mathbf{R} f$$

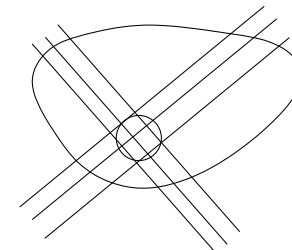
$\alpha = 0:$ $\Lambda_s = \mathcal{H} \frac{d}{ds},$ \mathcal{H} Hilbert transform

inversion formula for f is global



$\alpha = 1:$ $\Lambda_s^2 = -\frac{d^2}{ds^2},$ $\text{sing supp } \Lambda f \subset \text{sing supp } f$

inversion formula for Λf is local



Local tomography

f



Λf



Non-Uniqueness for discrete data

Smith et al. 1977: $s_i, i = 1, \dots, q, \vartheta_j, j = 1, \dots, p$

$$\exists f \neq 0 : \mathbf{R}f(s_i, \vartheta_j) = 0 \quad \forall i, j$$

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Natterer 1980: (s_i, ϑ_j) rectangular grid with $h = 2/q = \pi/p$

$$f \text{ ghost} \quad \Longrightarrow \quad \|f\|_{L^2} \lesssim h^\beta \|f\|_{H_0^\beta}, \quad \beta > 1/2$$

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Louis 1984: $0 < \tau < 1$

$$f \text{ ghost} \implies \int_{|\xi| \leq \tau(p-1)} |\widehat{f}(\xi)| \, d\xi \lesssim e^{-\lambda(\tau)p} \|f\|_{L^1}$$

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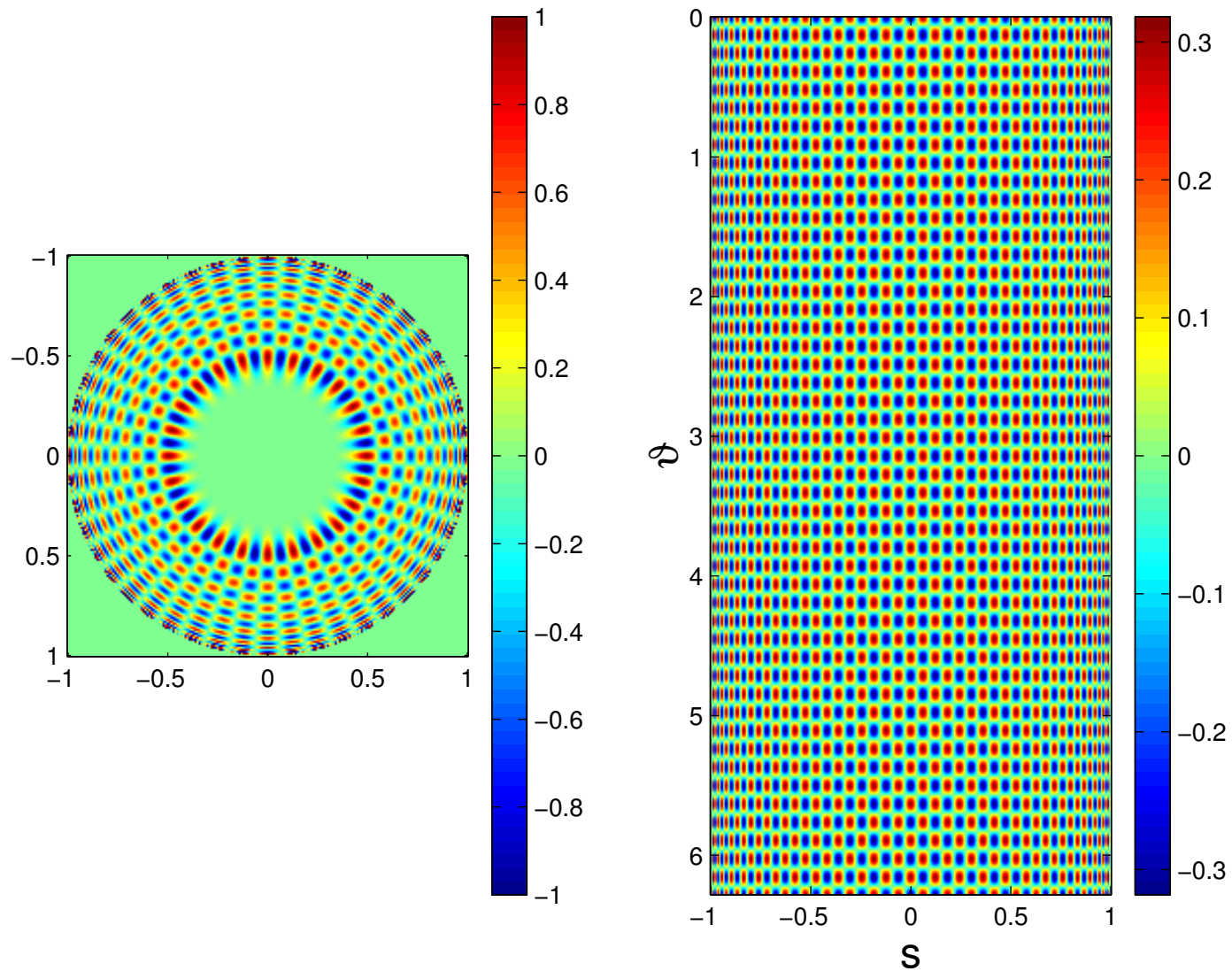
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Further analytical aspects: stability, sampling and resolution

CT ghost



Approximate inversion I

inversion formula: $f = \frac{1}{2\pi} \mathbf{R}^* \Lambda_s g \quad g = \mathbf{R}f$

approx. inversion: $f \star e = \mathbf{R}^* (v \star_s \mathbf{R}f), \quad e = \mathbf{R}^* v$

- e **mollifier** ($e \approx \delta$, centered about 0 with mean value 1)
- v **reconstruction filter/kernel**

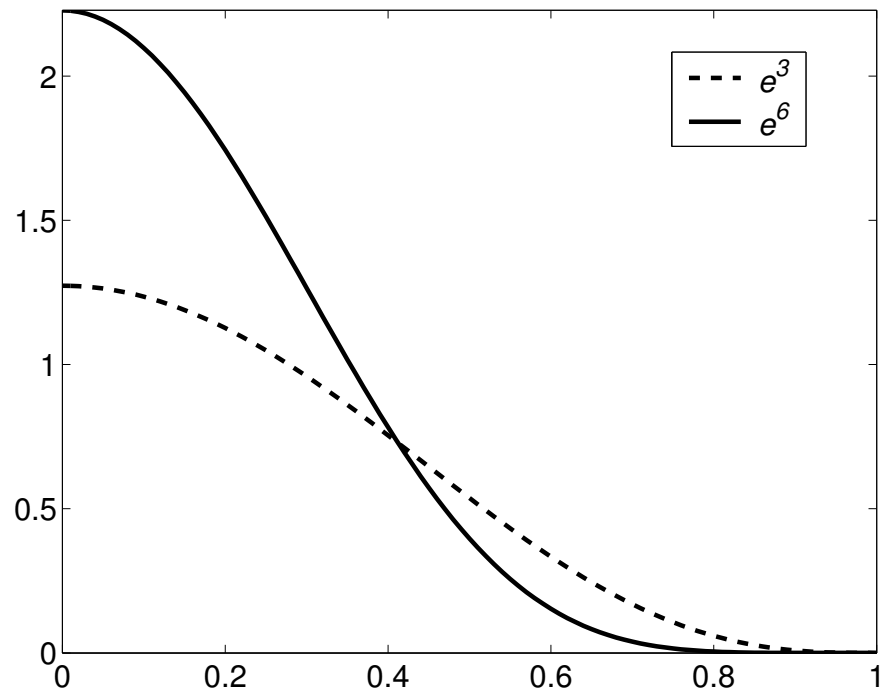
$$v = (2\pi)^{-1} \Lambda_s \mathbf{R}e \implies e = \mathbf{R}^* v$$

$$e_\gamma(x) = \gamma^{-2} e(x/\gamma), \quad v_\gamma(s) = \gamma^{-2} v(s/\gamma), \quad \gamma > 0$$

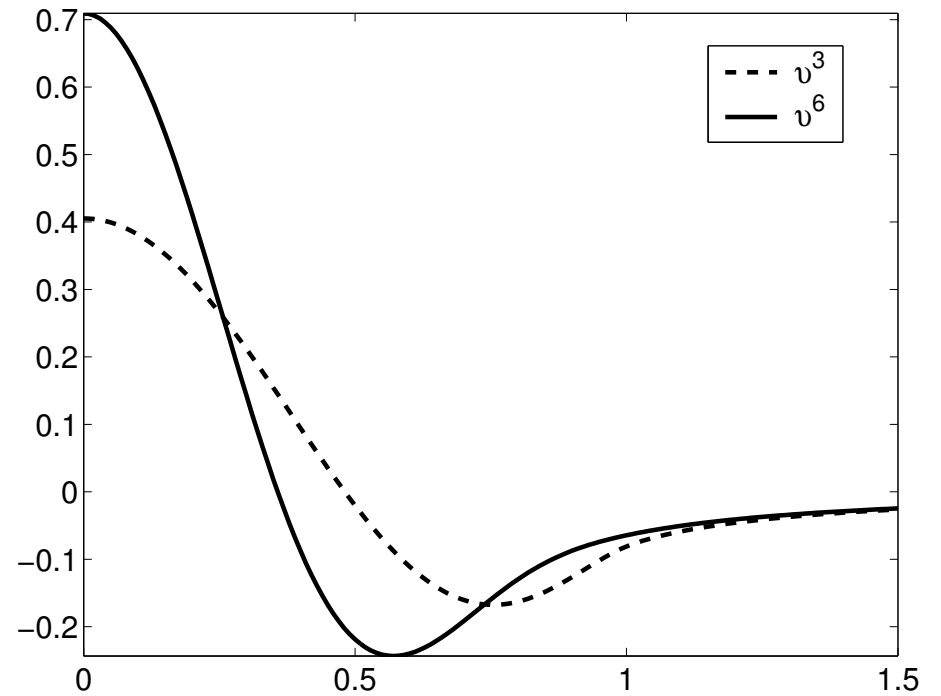
$$f \star e_\gamma \rightarrow f \text{ as } \gamma \rightarrow 0$$

Approximate inversion II

Mollifier



Rekonstruktionskerne zur Radon-Transformation



Reconstruction algorithm I

approx. inversion: $f \star e_\gamma = \mathbf{R}^* (v_\gamma \star_s g)$

discrete data: $g_{\ell,j} = \mathbf{R}f(\ell/q, j\pi/p)$, $\ell = -q, \dots, q$, $j = 0, \dots, p-1$

filtered backprojection: $f_R(x) = \mathbf{R}_p^* (v_\gamma \star_h g)(x)$

$$(v_\gamma \star_h g)_{k,j} = h \sum_{\ell=-q}^q v_\gamma(h(k-\ell)) g_{\ell,j}, \quad h = 1/q$$

How to choose γ in relation to h ?

Reconstruction algorithm II

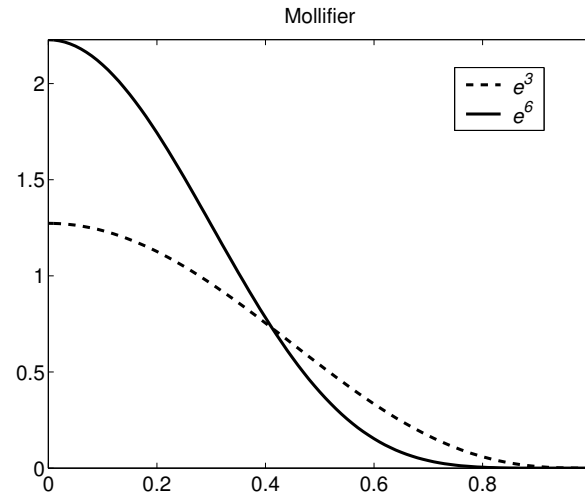
R. 2000: f essentially b -band-limited and $h \leq \pi/b$

$$f_R = f \star e_\gamma + m(\gamma, h) \Lambda^{-1} f + \text{discr. error}$$

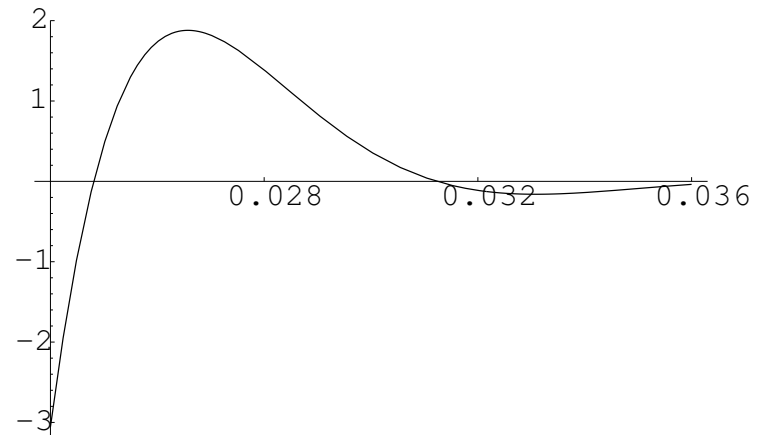
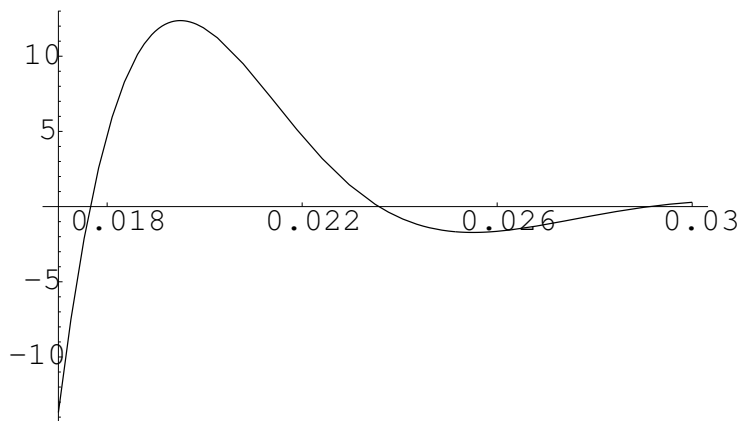
Strategy: Determine $\gamma = \gamma_h$ as a zero of $m(\cdot, h)$, that is,

$$m(\gamma_h, h) = 0$$

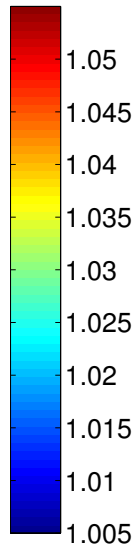
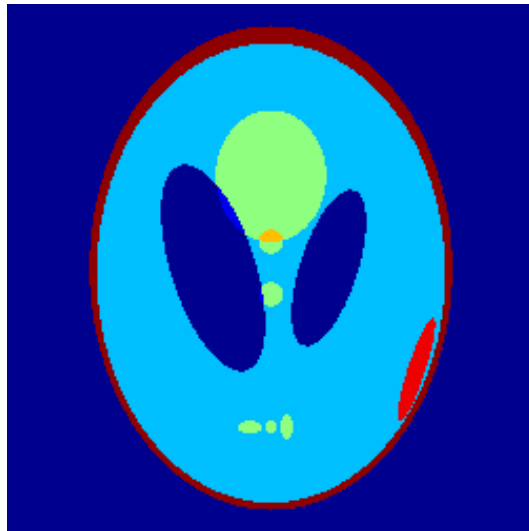
Reconstruction algorithm III



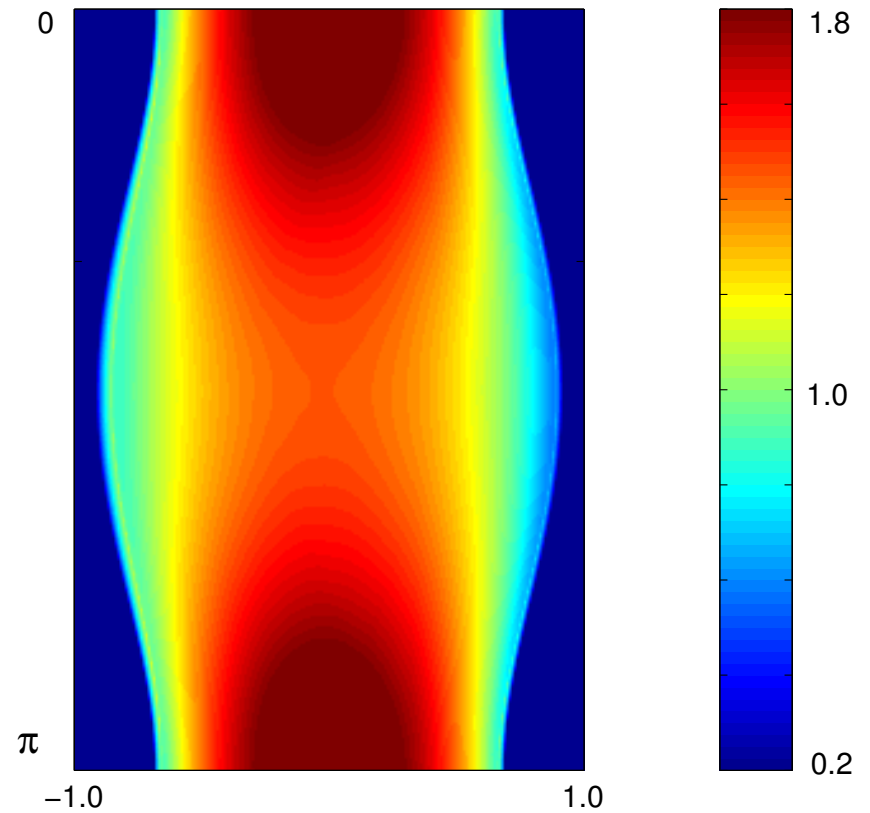
$$h = 0.01$$



Shepp-Logan head phantom

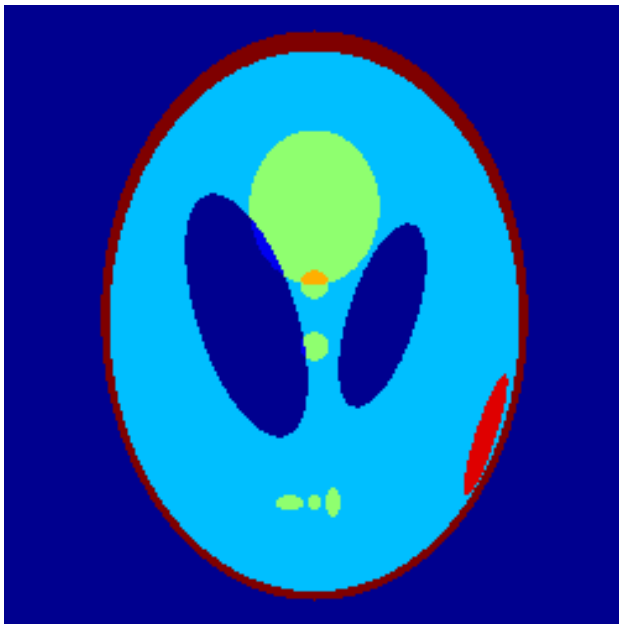


Tomographic Data

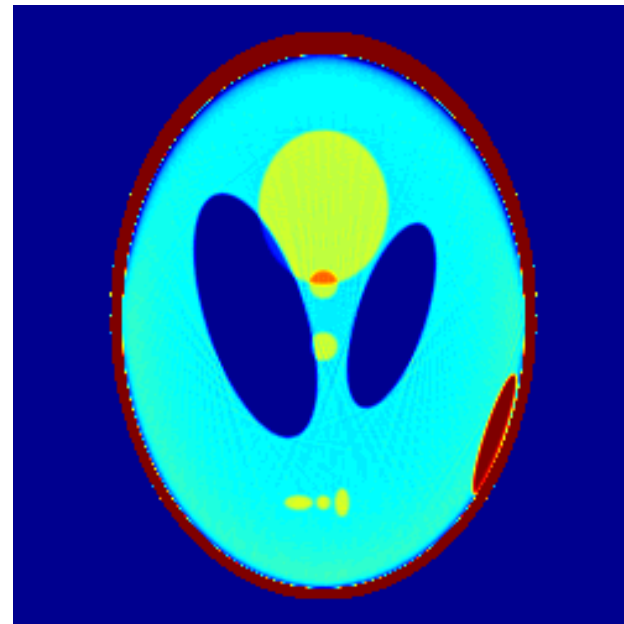


Reconstructions I

$$e(x) = \begin{cases} (1 - \|x\|^2)^6 & : \|x\| \leq 1 \\ 0 & : \text{otherwise} \end{cases}, \quad h = 0.01$$

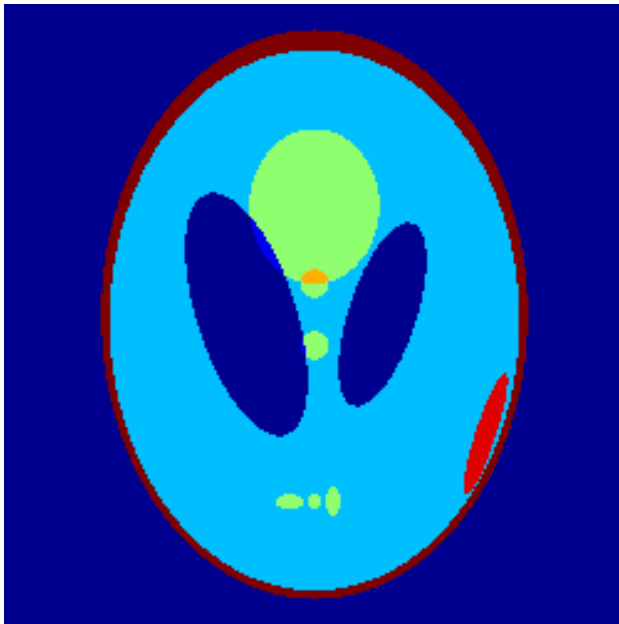


original

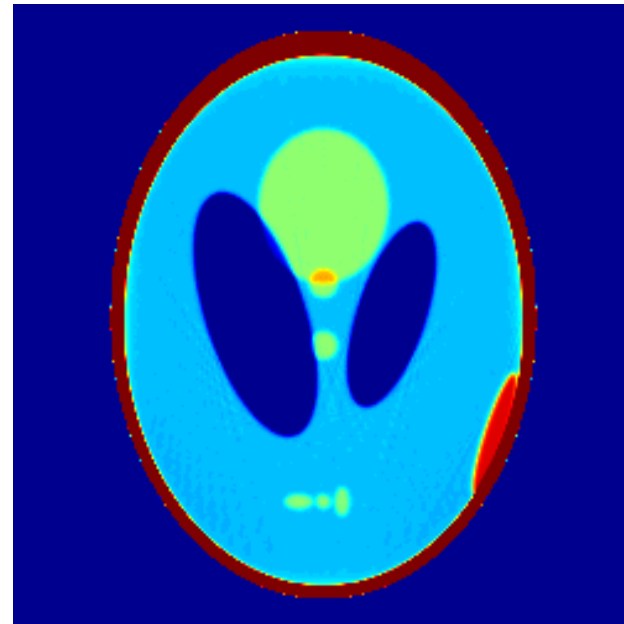


$\gamma = 0.01765299..$
(sm. zero of m),
rel. ℓ^2 -error: 0.0816

Reconstructions II

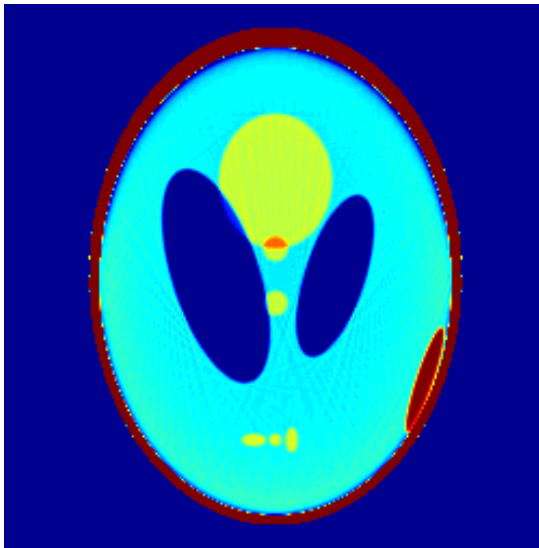


original

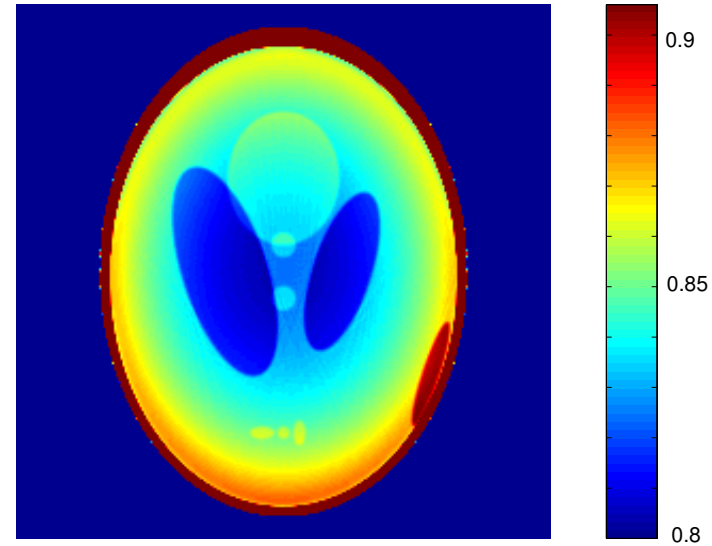


$\gamma = 0.02357177..$
(2nd sm. zero of m),
rel. ℓ^2 -error: 0.1001

Violating the zero condition

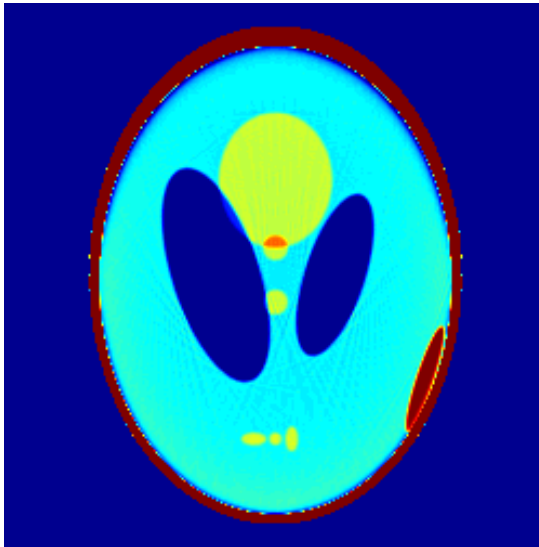


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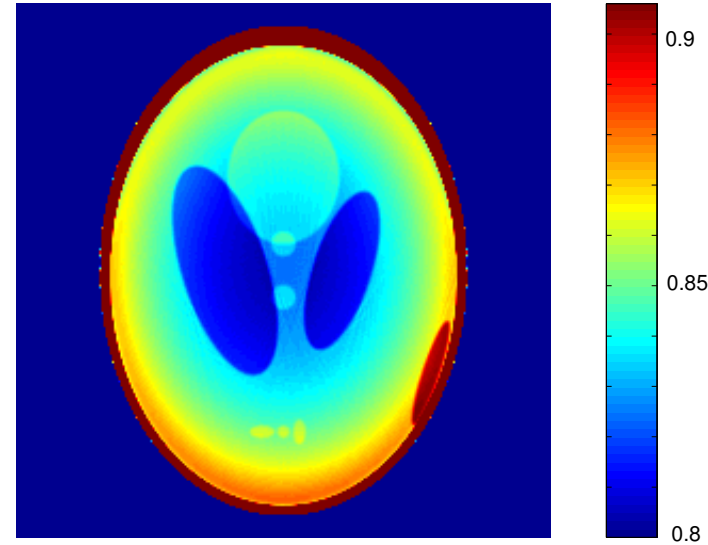


$\gamma = 0.0177..$
rel. ℓ^2 -error: 0.1730

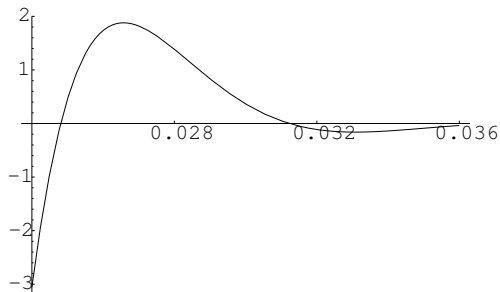
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