Augmented Lagrangian Methods for Perfect Plasticity

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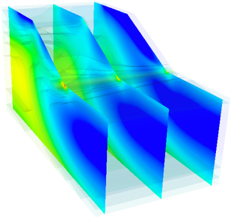

- AFRICOMP -

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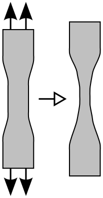
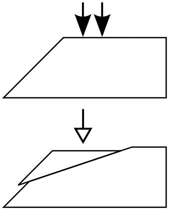
Elasto-Plasticity

Metals:
(Tension test)



Soils:
(Slope failure)



- Irreversible deformation, path-dependent.
- Plastic deformation may be caused by different physical effects:
  - Metals: crystallographic defects (dislocations)
  - Soils (granular materials): Interaction of particles (friction)
- Unified modeling possible in the sense of continuum mechanics.

Time-discrete Elastoplasticity

- $\Omega \subset \mathbb{R}^d$, $\partial\Omega = \Gamma_D \cup \Gamma_N$ (disjoint).
- Stress tensor $\boldsymbol{\sigma} : \Omega \to S = \{\boldsymbol{\eta} \in \mathbb{R}^{d,d} : \boldsymbol{\eta} = \boldsymbol{\eta}^T\}$.
- Displacement $u : \Omega \to \mathbb{R}^d$.
- Path-dependency: plastic strain tensor $\varepsilon_p^{old} : \Omega \to S$.

$$
\begin{aligned}
-\operatorname{div} \sigma(x) &= b(x)\,, \\
\sigma(x) &= \mathbb{C}[\nabla^S u(x) - \varepsilon_p(x)]\,, \\
u(x) &= 0 \quad \text{on } \Gamma_D\,, \qquad \sigma(x)\nu(x) = t_N(x) \quad \text{on } \Gamma_N\,.
\end{aligned}
$$

Elasticity tensor $\mathbb{C} \in \mathcal{L}(S, S)$, e.g.: $\mathbb{C}[\varepsilon] = 2\mu\varepsilon + \lambda \operatorname{tr}(\varepsilon)\boldsymbol{I}$.

Time-discrete evolution: associated flow rule/ normality condition

$$\varepsilon_p(x) - \varepsilon_p^{old}(x) \in N_K(\sigma(x))$$

with N_K being the normal cone to a convex set $K \subset S$.

Time-discrete Elastoplasticity

If $K = \{\eta \in S : f_i(\eta) \le 0, i = 1, \dots, P\}$ with convex yield functions $f_i : S \to \mathbb{R}$, then

$$\varepsilon_p(x) - \varepsilon_p^{old}(x) = \sum_{i=1}^{P} \lambda_i(x)\, Df_i(\sigma(x))\,.$$

The plastic multipliers (Lagrange) $\lambda_i : \Omega \to \mathbb{R}$ are determined by the complementarity (Karush-Kuhn-Tucker) conditions

$$\lambda_i(x) \ge 0\,, \qquad f_i(\sigma(x)) \le 0\,, \qquad \lambda_i(x) \cdot f_i(\sigma(x)) = 0\,. \tag{1}$$

Substitution and the equilibrium condition in weak form give $(V = \{v : \Omega \to \mathbb{R}^d : v|_{\Gamma_D} = 0\})$

$$0 = \mathbb{C}^{-1}[\sigma(x)] - \left(\nabla^S u(x) - \varepsilon_p^{old}(x)\right) + \sum_{i=1}^{P} \lambda_i(x)\, Df_i(\sigma(x))\,, \tag{2}$$

$$0 = \int_\Omega \sigma(x) : \nabla^S v(x) - \ell(v)\,, \qquad v \in V\,, \tag{3}$$

with the load functional $\ell(v) = \int_\Omega b \cdot v\, dx + \int_{\Gamma_N} t_N \cdot v\, da$.

The minimization problem

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Some notation:

- $\boldsymbol{S} = L^2(\Omega, S)$,
- $\boldsymbol{E} = \{\sigma \in \boldsymbol{S} : -\operatorname{div}\sigma = b \text{ in } \Omega,\ \sigma\nu = t_N \text{ on } \Gamma_{\mathrm{N}}\}$,
- $\boldsymbol{K} = \{\sigma \in \boldsymbol{S} : \sigma(x) \in K \text{ a.e.}\}$, Indicator-Fct. $\chi_{\boldsymbol{K}}(\sigma) = \begin{cases} 0 & , \sigma \in \boldsymbol{K} , \\ \infty & , \sigma \notin \boldsymbol{K} . \end{cases}$

Then, (1), (2) and (3) are the optimality conditions (Karush-Kuhn-Tucker) of the constrained minimization problem

$$(D) \quad \text{Minimize} \quad J(\sigma) \quad \text{subject to} \quad \sigma \in \boldsymbol{E} \cap \boldsymbol{K} .$$

with $J(\sigma) = \frac{1}{2}(\mathbb{C}^{-1}[\sigma], \sigma)_{\boldsymbol{S}} + (\sigma, \varepsilon_p^{old})_{\boldsymbol{S}}$, (uniformly convex).
Alternativly this can be written as

$$(D) \quad \text{Minimize} \quad J(\sigma) + \chi_{\boldsymbol{K}}(\sigma) \quad \text{subject to} \quad \sigma \in \boldsymbol{E} .$$

Convex duality in perfect plasticity

Consider the so-called dual problem

$$(D) \quad \text{Minimize} \quad J(\sigma) + \chi_{\boldsymbol{K}}(\sigma) \quad \text{subject to} \quad \sigma \in \boldsymbol{E}\,.$$

With the Lagrangian

$$L(\sigma, u) = J(\sigma) + \chi_{\boldsymbol{K}}(\sigma) - \int_\Omega \sigma(x) : \nabla^{\mathrm{S}} u(x) + \ell(u)\,.$$

we find

$$(D) \quad \Longleftrightarrow \quad \text{Minimize} \quad \sup_{u \in V} L(\sigma, u)$$

and the corresponding primal problem

$$(P) \quad \text{Maximize} \quad \mathcal{E}(u) = \inf_{\sigma \in \boldsymbol{S}} L(\sigma, u) = L\Big(P_K(\mathbb{C}[\nabla^{\mathrm{S}} u - \varepsilon_p^{old}]), u\Big)\,.$$

$P_K : \boldsymbol{S} \to \boldsymbol{K}$ is the orthogonal projection onto $\boldsymbol{K}$ w.r.t. $\mathbb{C}^{-1}$.

A few remarks ...

What we know ...

- The dual problem (D) has a unique solution if the safe-load condition (Slater condition) is satisfied.
- The primal functional $\mathcal{E}$ has only linear growth. A solution only exists in the measure space $BD(\Omega)$ (bounded deformation).
- Solving (P) with a (generalized) Newton method yields the classical radial return method / closest point projection.

Some known problems ...

- Primal problem is not uniformly concave/convex: generalized Newton methods (radial return) often exhibit bad global convergence properties. Situation becomes worse with ongoing mesh refinement: mesh-dependence.
- For realistic parameters, the same is true for hardening plasticity and viscoplasticity.

The Augmented Lagrange Regularization

Question: How to improve global convergence properties while retaining fast local convergence?

One possibility: Augmented Lagrange regularization/approximation. Introduced by Ito/Kunisch (also called generalized Moreau-Yosida-approximation).

For $\boldsymbol{\delta} \in \boldsymbol{S}$ (arbitrary at the moment) and $\alpha > 0$, set

$$\chi^\alpha_{\boldsymbol{K}}(\boldsymbol{\sigma}, \boldsymbol{\delta}) = \inf_{\boldsymbol{\theta} \in \boldsymbol{S}} \left\{ \chi_{\boldsymbol{K}}(\boldsymbol{\sigma} - \boldsymbol{\theta}) + (\boldsymbol{\delta}, \boldsymbol{\theta})_{\boldsymbol{P}} + \frac{\alpha}{2} \|\boldsymbol{\theta}\|^2_{\mathbb{C}^{-1}} \right\} = \dots$$
$$= \frac{\alpha}{2} \Big\| \sigma + \frac{1}{\alpha}\mathbb{C}[\delta] - P_K\Big(\sigma + \frac{1}{\alpha}\mathbb{C}[\delta]\Big) \Big\|^2_{\mathbb{C}^{-1}} - \frac{1}{2\alpha} \Big\| \mathbb{C}[\delta] \Big\|^2_{\mathbb{C}^{-1}} .$$

- Setting $\boldsymbol{\delta} = 0$ gives the “classical” Moreau-Yosida-approximation, i.e. $\chi^\alpha_{\boldsymbol{K}}(\sigma) = \frac{\alpha}{2} \|\sigma - P_K(\sigma)\|^2$, which is the viscoplastic regularization of Duvaut/Lions, also used in the existence proof of Johnson.
- For $\alpha \to \infty$, we have $\chi^\alpha_{\boldsymbol{K}}(\sigma, \boldsymbol{\delta}) \to \chi_{\boldsymbol{K}}(\sigma)$ for all $\boldsymbol{\delta} \in \boldsymbol{S}$.

Approximation of perfect plasticity

Regularized dual problem (δ is fixed)

$$(D_\alpha) \quad \text{Minimize} \quad J(\sigma) + \chi^\alpha_{\boldsymbol{K}}(\sigma, \delta) \qquad \text{subject to} \quad \sigma \in \boldsymbol{E}\,.$$

Based on the Augmented Lagrangian

$$L^\alpha(\sigma, u, \delta) = J(\sigma) + \chi^\alpha_{\boldsymbol{K}}(\sigma, \delta) - \int_\Omega \sigma(x) : \nabla^{\mathrm{S}} u(x) + \ell(u)\,,$$

the corresponding primal functional $\mathcal{E}^\alpha$ is uniformly concave, and admits a unique solution in standard Sobolev spaces.

Theorem (S.)

Let the safe-load-condition be satified and let $\hat{\sigma} \in \boldsymbol{E} \cap \boldsymbol{K}$ solve the dual problem (D) of perfect plasticity. Moreover, let $\delta \in \boldsymbol{S}$ be arbitrary but fixed and by $\sigma_\alpha \in \boldsymbol{E}$, we denote the solution of the regularized problem (D_α) with parameter $\alpha > 0$.

Then: $\quad \lim_{\alpha \to \infty} \left\| \sigma_\alpha - \hat{\sigma} \right\|_{\boldsymbol{S}} = 0\,.$

Approximation of perfect plasticity

Discussion of the approximation result ...

- Approximation result requires $\alpha \to \infty$.
- Result is well-known for $\delta = 0$.
- Desireable: approximation result for finite $\alpha > 0$.

Observation for perfect plasticity: if the plastic strain $\hat{\varepsilon}_p = \nabla^S \hat{u} - \mathbb{C}^{-1}[\hat{\sigma}]$ is contained in $\boldsymbol{S}$, then we have

$$\hat{\varepsilon}_p = D_\sigma \chi^\alpha_{\boldsymbol{K}}(\hat{\sigma}, \hat{\varepsilon}_p)\,.$$

Idea: Instead of keeping δ fixed, update δ during the approximation procedure.

Augmented Lagrange Method (ALM)

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ALM algorithm (may be extended by an additional α-Update)

S0) Choose $\varepsilon_{p,\alpha}^{0} \in \boldsymbol{S}$ and $\alpha > 0$. Set $k := 1$.

S1) Determine primal-dual solution $(\sigma_\alpha^k, u_\alpha^k) \in \boldsymbol{E} \times V$ of (D_α) and (P_α) with $\delta = \varepsilon_{p,\alpha}^{k-1}$.

S2) Update $\varepsilon_{p,\alpha}^{k} = D_\sigma \chi_{\boldsymbol{K}}^{\alpha}(\sigma_\alpha^k, \varepsilon_{p,\alpha}^{k-1})$.

S3) Check for convergence, $k := k + 1$, go to S1).

Theorem (S.)

Let the safe-load-condition be satisfied and let $\hat{\sigma} \in \boldsymbol{K} \cap \boldsymbol{E}$ solve the dual problem (D) of perfect plasticity. Additionally assume $\hat{\varepsilon}_p \in \boldsymbol{S}$.

Then, for fixed $\alpha > 0$: $$\sum_{k=1}^{\infty} \left\| \sigma_\alpha^k - \hat{\sigma} \right\|_{\boldsymbol{S}}^2 \leq \frac{1}{2\alpha} \left\| \mathbb{C}[\varepsilon_{p,\alpha}^{0} - \hat{\varepsilon}_p] \right\|_{\boldsymbol{S}}^2 .$$

A few remarks on ALM

The solution $(\hat{\sigma}, \hat{u}, \hat{\varepsilon}_p)$ of the perfect plasticity problem can be characterized by

$$0 = D_\sigma L^\alpha(\hat{\sigma}, \hat{u}, \hat{\varepsilon}_p)\,, \tag{4}$$

$$0 = D_u L^\alpha(\hat{\sigma}, \hat{u}, \hat{\varepsilon}_p)\,, \tag{5}$$

$$0 = D_{\varepsilon_p} L^\alpha(\hat{\sigma}, \hat{u}, \hat{\varepsilon}_p)\,. \tag{6}$$

These three conditions replace (1), (2) and (3) as the optimality system. Note that the plastic multiplier λ does not appear in the current situation, but only the plastic strain (increment). However, λ is hidden in the evaluation of the projection P_K.

Then, the ALM is just the iteration of

- Determine $(\sigma_\alpha^k, u_\alpha^k)$ via (4), (5) for given $\varepsilon_{p,\alpha}^{k-1}$.
- Determine ε_p^k via (6) for given $(\sigma_\alpha^k, u_\alpha^k)$ and $\varepsilon_{p,\alpha}^{k-1}$.

This results in a first order method for fixed $\alpha > 0$. If additionally $\alpha \to \infty$, then ALM converges superlinearly.

Second order iterations

Potentially faster algorithm can be derived by applying a generalized Newton method to (4), (5) and (6) directly.

One can distinguish mainly two cases:

- Since (4) and (6) are local equations (pointwise a.e. in Ω), a nonlinear Schur reduction can be used to eliminate σ and ε_p, i.e. it is possible to determine $\sigma \equiv \sigma(u)$ and $\varepsilon_p \equiv \varepsilon_p(u)$. Substituting into (5), the radial return method is re-obtained.
- Applying a (generalized) Newton method to all equations $DL^\alpha(\cdot) = 0$ leads to an active set type algorithm. Superlinear convergence can be shown in the discrete setting.

Solution procedure - parallel FEM

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The regularized dual problem (D_α) is solved via its corresponding primal problem (P_α) which is uniformly convex with modulus $\frac{1}{1+\alpha}$.
(P_α) is solved via a generalized Newton method.

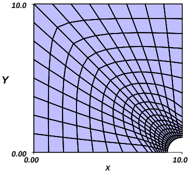
Implementation: parallel FEM-suite M++ (Wieners).

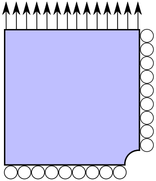
- Linear systems are solved by a Multigrid preconditioner.
- Non-overlapping parallel domain decomposition.
- Parallelization is based on MPI.
- Block-Gauss-Seidel smoothing. Local operation: no communication.
- Standard Lagrange finite elements for the displacement.
- Stress/strain is evaluated at Gaussian quadrature points.

M++ can do a lot more than "just" plasticity: Maxwell, Finite Elasticity, (Navier-) Stokes, BEM, Eigenvalue problems, DG for conservation laws. Parallel linear solvers: "self-made" and interfaces to library solvers (Mumps, SuperLU, HB, etc.).

Newton vs. Augmented Lagrange (ALM)

- Von Mises perfect plasticity (von Mises).
  $f(\sigma) = |\mathrm{dev}\,\sigma| - K_0$
- Traction b.c. on the upper boundary
  Symmetric Dirichlet b.c.
- Newton/Radial Return: Backtracking Line Search.
- ALM: Additional α-Update.



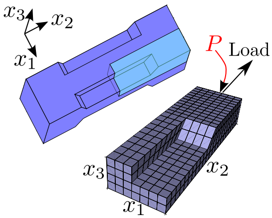


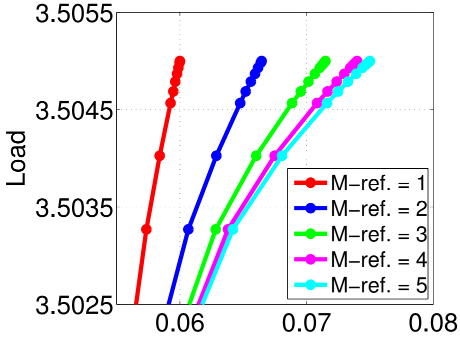
Mesh-Ref.	Newton Iter.	ALM Iter.
3	9	5 (9 LinS)
4	13	5 (10 LinS)
5	22	5 (10 LinS)
6	38	5 (11 LinS)
7	70	6 (11 LinS)

Mesh-Refinement	3	4	5	6	7
Unknown w.r.t. u	33 282	132 098	526 338	2 101 250	8 396 802

Augmented Lagrange / Mesh convergence

- Von Mises perfect plasticity.
- Load-Displacement-Cuve at *P*.
- Displacement seems to converge pointwise.





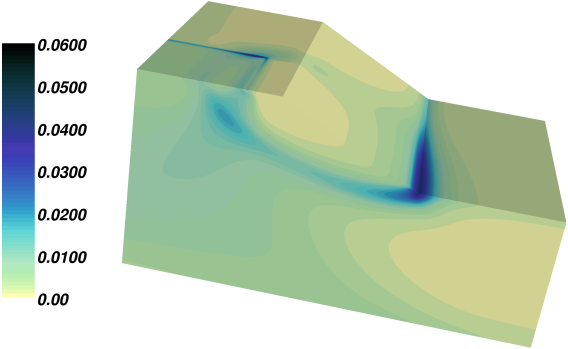
ALM Iter.	Error
0	1.6e-02
1	4.8e-00
2	9.0e-01
3	3.2e-02
4	3.4e-04
5	1.0e-06
6	2.2e-08
7	1.9e-09

Mesh-Refinement	1	2	3	4	5
Unknowns w.r.t. u	16 305	116 271	875 931	6 795 315	53 523 555

ALM not limited to von Mises plasticity

Augmented Lagrangian methods for perfect plasticity can be used for any associated material law (including multi-yield).

Drucker-Prager plasticity: $f(\sigma) = |\operatorname{dev}\sigma| + M\operatorname{tr}(\sigma) - C$



Accumulated plastic strain. 50 973 123 Unknowns w.r.t. u on 512 processors.

All computations performed with the parallel FEM-Suite M++ (Wieners)

Key take-aways

- Be careful with the results of your plasticity FEM computation. Even for simple geometries, a lot of DoFs are necessary to obtain mesh-convergence. Whenever possible: use adaptivity.
- In large scale computations, standard algorithms are not always satisfying (e.g. radial return) w.r.t. global convergence.
- Globalization techniques are at hand from (numerical) optimization.
- (Iterated) regularization (can be interpreted as a homotopy approach) has potentially better global properties.
- Fast local convergence is still possible.
- Examination of nonlinear solvers in function space is helpful for developing robust methods.

Thank you for your attention!

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