Karlsruhe Institute of Technology
Institute for Algebra and Geometry
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Karlsruhe, January 12, 2010

Student Nr.:

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Worksheet No.10
Advanced Mathematics I

Exercise 46: Consider the the following power series:

$$f(x) = \sum_{k=0}^{\infty} \frac{x^{2k}}{k!}, \qquad g(x) = \sum_{k=0}^{\infty} (k+1)x^k .$$

Determine the domains, in which the derivatives of the functions can be determined by differentiating the corresponding power series, and compute these derivatives.

Exercise 47: Consider the function $f(x) = x^5$, $x \in \mathbb{R}$. Show that:

(a) f has an inverse function g (sketch!).

(b) g is differentiable for $x \neq 0$, but not for $x = 0$.

(c) Give the derivative $g'(x)$ for $x \neq 0$.

Exercise 48: Give the domain $D \subseteq \mathbb{R}$ for (a) and (b) and compute the derivative of the following real-valued functions:

(a) $f(x) = e^x \cdot \cos^2(x) \cdot (\cos x + 3 \sin x)$,

(b) $f(x) = \left(\frac{x+1}{x-1}\right)^2$,

(c) $f(x) = a^{(x^x)}, \quad a > 0, x \in \mathbb{R}_{>0}$,

(d) $f(x) = x^{(a^x)}, \quad a > 0, x \in \mathbb{R}_{>0}$.

Exercise 49: Consider the function

$$f(x) = \begin{cases} e^x & x \leq 0 \\ \cos(x) + x & 0 < x \end{cases} .$$

(a) Show that f is continuously differentiable arbitrarily many times at $x \neq 0$.

(b) Show that f is (once) continuously differentiable at $x = 0$.

(c) Is f twice continuously differentiable at $x = 0$?

Exercise 50: The inverse function of the tangent is the function

$$\arctan : \mathbb{R} \longrightarrow \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

(a) Compute the derivative of arctan using the formula for computing the derivative of an inverse.

(b) arctan can be expanded in power series for $|x| < 1$:

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}.$$

Compute $(\arctan x)'$ by differentiating this power series.

Due date: Please hand in your homework on Thursday, January 21, 12:00, into the AM1-box near Seminar room 1C-03, Allianz-Gebäude (05.20).

Tutorial 10
Advanced Mathematics I

Exercise T37: Prove that the function $f : [0, \infty) \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x & , \quad x \in [0,1] \ , \\ 2x - 1 & , \quad x \in (1, \infty) \ , \end{cases}$$

(a) is continuous for $0 \leq x < \infty$,

(b) differentiable for $0 < x < 1$ and $1 < x < \infty$ and

(c) not differentiable at $x = 1$.

Exercise T38: Give the domain $D \subseteq \mathbb{R}$ and compute the derivative of the following real-valued functions:

(a) $f(x) = \dfrac{x}{x^2+1}$, (b) $f(x) = \dfrac{\cos^2 x}{1+\cot x} + \dfrac{\sin^2 x}{1 + \tan x}$,

(c) $f(x) = (\sqrt{x}+1) \cdot \left(\frac{1}{\sqrt{x}} - 1 \right)$, (d) $f(x) = \cos^2 x \cdot \cos(x^2)$.

Exercise T39: Let

$$f(x) = \frac{1}{2} \ln \left(\frac{x+1}{x-1} \right), \qquad x > 1.$$

(a) Compute the derivative of f.

(b) f can also be represented by the following power series:

$$f(x) = \sum_{k=1}^{\infty} \frac{1}{(2k+1)\, x^{2k+1}}.$$

Use this representation to compute the derivative of f.

Exercise T40:

(a) Give the interval $I \subset \mathbb{R}$, where the function $f(x) = \cosh x$ is strictly monotonic increasing and thus invertible.

(b) Apply the representation $\cosh x = \frac{1}{2}(e^x + e^{-x})$ to determine the inverse function $f^{-1}(x) = \operatorname{Arcosh}(x)$ and its derivative $(f^{-1})'(x)$.

Tutorial date: Tuesday, January 19, 2010, 3:45-5:15 pm.