Karlsruhe Institute of Technology
Institute for Algebra and Geometry
Dr. T. Arens
Dipl.-Math.techn. A. Schkarbanenko
Dipl.-Math. S. Heumann

51	52	53	54	55	Σ

Karlsruhe, January 19, 2010

Student Nr.:

....

Worksheet No.11
Advanced Mathematics I

Exercise 51: Use the mean value theorem to prove the following subtasks.

(a) Prove the inequality

$$\ln(1+x) \leq \frac{x}{\sqrt{1+x}} \qquad \text{for } x > 0\,.$$

Hint: Consider the function $f(t) = \ln(1+t) - \frac{t}{\sqrt{1+t}}$ in the interval $[0, x]$.

(b) Show the estimations

$$1 - \frac{1}{x} < \ln x < x - 1, \quad x \in (1, \infty).$$

Can you give an upper and lower bound to the real number $a = 2\ln 3 - 3\ln 2$ using these inequalities?
Hint: Consider the function $f(t) = \ln t$ for $t \in (1, x)$.

(c) Show the Lipschitz continuity of the function

$$f(x) = \sqrt{1+x}\,, \qquad 0 \leq x < 3\,,$$

and compute its Lipschitz constant.

Exercise 52: Compute the limits

(a) $\lim\limits_{x\to 0} \dfrac{\cos x + 3x - 1}{2x}$, (b) $\lim\limits_{x\to a} \dfrac{x^a - a^x}{a^x - a^a}$, $a > 0, a \neq 1$, (c) $\lim\limits_{x\to\infty} \dfrac{2\ln x}{x^b}$, $b > 0$, (d) $\lim\limits_{x\to\infty} \dfrac{\ln(\ln x)}{\ln x}$.

Exercise 53: Consider $f(x) = \sqrt[3]{2x+2}$, where $x \geq -1$.

(a) Give the Taylor polynomial of f of degree 2 with center of expansion $x_0 = 3$.

(b) Estimate for $x \in [-\frac{1}{2}, 3]$ the Lagrange form of the Taylor polynomial of 2-nd degree of f independent of x.

Exercise 54: Let the function $f\colon [-1,2) \cup (2,3] \to \mathbb{R}$ be defined by

$$f(x) = \frac{x^3 - 4x^2 + x + 6}{x - 2}.$$

Show that $f(x) \leq 0$ is valid for all $x \in [-1,2) \cup (2,3]$.

Exercise 55: Between Karlsruhe and Bruchsal the suburban train S3 runs with a maximum speed of 160 km/h. The power consumption of such a train is proportional to the square of its velocity. Powering the train at a speed of 50 km/h costs 100 EUR per hour. On top of that there are fixed costs of 400 EUR per hour for the operation of the train (labour costs, maintenance etc.).

(a) At which speed are the operation costs per kilometer minimal?

(b) The mean daily revenue is 14 EUR per kilometer. How fast should the train go to maximize the total profit (i.e. the difference between revenue and costs) per hour?

Due date: Please hand in your homework on Thursday, January 28, 12:00, into the AM1-box near Seminar room 1C-03, Allianz-Gebäude (05.20).

Tutorial 11
Advanced Mathematics I

Exercise T41: Use the mean value theorem to prove

(a) the inequality

$$|\cos e^x - \cos e^y| \leq |x - y|$$

for $x, y \leq 0$,

(b) the Lipschitz continuity of the fuction $f(x) = \sin x$ for $x \in \mathbb{R}$,

(c) the limit

$$\lim_{n\to\infty} (\sqrt[3]{n^2 + k^2} - \sqrt[3]{n^2}) = 0.$$

Hint: Consider the function $x \mapsto \sqrt[3]{x}$.

Exercise T42: Compute the given limits. Why is it not allowed to use de l'Hôpital's rule in part (c)?

(a) $\displaystyle\lim_{x\to 0} \frac{\tan x - x}{x^3}$ (b) $\displaystyle\lim_{x\to\infty} \frac{x^2 e^x}{(e^x - 1)^2}$ (c) $\displaystyle\lim_{x\to\infty} \frac{x - \sin x}{x + \sin x}$

Exercise T43: Calculate all the derivatives $f^{(n)}$, $n = 0, 1, 2, \ldots$ of the function f and give the Taylor series for f with center of expansion $x_0 = 0$. Where does the series in part (a) converge?

(a) $f(x) = \cosh \dfrac{x}{2}, \quad x \in \mathbb{R},$ (b) $f(x) = \sqrt{1 + x}, \quad |x| \leq 1.$

Hint for (b): The derivatives of f have the following form: $f^{(k)}(x) = -(-1)^k \frac{(2k-2)!}{2^{2k-1}\,(k-1)!}(1 + x)^{-\frac{2k-1}{2}}$.

Exercise T44: Show that for $x \in [-1, 1]$

$$-1 + x - \frac{x^2 - 1}{x^2 + 1} \leq \arctan(x) \leq 1 + x + \frac{x^2 - 1}{x^2 + 1}.$$

Hint: Analyze the extrema of the terms' differences.

Tutorial date: Tuesday, January 26, 2010, 3:45-5:15 pm.