Karlsruhe Institute of Technology
Institute for Algebra and Geometry
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Karlsruhe, November 10, 2009

Student Nr.:

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Worksheet No.3
Advanced Mathematics I

Exercise 11: Consider the sequence $(a_n)_n$ with

$$a_n = \frac{n(n+3)-4}{n^2-1}, \qquad n \in \mathbb{N}.$$

Investigate the convergence computing a sequence index N such that $|a_n - 1| < \varepsilon$ for all $n \geq N$, if

$$\text{(i) } \varepsilon = \frac{1}{10}, \qquad \text{(ii) } \varepsilon = \frac{1}{100}, \qquad \text{(iii) } \varepsilon > 0 \text{ arbitrary.}$$

Does the sequence $(a_n)_n$ converge? If so, which limit does she have?

Exercise 12: Determine the limit of the sequence $(a_n)_n$, where

$$\text{(a)} \quad a_n = \sqrt[n]{4 + \frac{n-1}{n+1}}, \quad \text{(b)} \quad a_n = \frac{n^4-2}{n^2+4} + \frac{n^3(3-n^2)}{n^3+1}, \quad \text{(c)} \quad a_n = \sqrt[n]{34^n + 118^n} \cdot \left[\frac{(n+4)^4}{n^3} - n + 1\right] + 3\,.$$

Exercise 13: Prove the sequences given by

$$\text{(a)} \quad a_n = \sqrt{q^n + n} - \sqrt{n} \quad \text{with a fixed } q > 0, \qquad \text{(b)} \quad b_n = \frac{2}{n+1} \prod_{k=1}^{n} \left(1 + \frac{2}{k}\right) - \sqrt{n^2+1},$$

for the convergence behaviour, i.e. convergence or divergence. Compute the limit in case of convergence.
Hint for part (b): Prove by mathematical induction that for $p_n := \frac{2}{n+1} \prod_{k=1}^{n} \left(1 + \frac{2}{k}\right)$ it holds: $p_n = n + 2$.

Exercise 14: Compute the limits of the following complex sequences

$$\text{(a)} \quad a_n = 2 + \frac{4i^n}{n} + \left(\frac{1}{4} + \frac{1}{2}i\right)^n, \qquad \text{(b)} \quad b_n = \frac{n^3 + (in^2+1)(6+in)}{\frac{1}{n}\sum_{k=1}^{n} ik^2}$$

Exercise 15: Let the sequence $(a_n)_n$ be defined as $a_0 = 0$, $a_1 = 1$, $a_{n+1} := \frac{1}{2}(a_n + a_{n-1})$, $n \in \mathbb{N}$. Furthermore, let the sequence $(c_n)_n$, $n \in \mathbb{N}$ be defined by $c_n := a_n - a_{n-1}$.

(a) Give an explicit formula for c_n.

(b) Show that for all $n \in \mathbb{N}$ it holds $a_{2n} < a_{2n+1}$.

(c) Express a_n using $c_1, c_2, \ldots$ and prove that $\lim\limits_{n\to\infty} a_n = \frac{2}{3}$.

Due date: Please hand in your homework on Thursday, November 19, 12:00, into the AM1-box near Seminar room 1C-03, Allianz-Gebäude (05.20).

Tutorial 3
Advanced Mathematics I

Exercise T9: Consider the sequence $(a_n)_n$ with $a_n = \frac{n-1}{n+1}$, $n \in \mathbb{N}$. Determine an index N such that $|a_n - 1| < \varepsilon$ for every $n \geq N$, when

(a) $\varepsilon = \dfrac{1}{10}$, (b) $\varepsilon = \dfrac{1}{1000}$, (c) $\varepsilon > 0$ is arbitrary.

(d) Does the sequence $(a_n)_n$ converge? If so, to what limit?

Exercise T10: Compute the limits of the following sequences applying the limit rules

(a) $a_n = \dfrac{1+n+n^2}{n(n+1)}$, (c) $c_n = \sqrt{n^2+an+b} - n$, $a, b \in \mathbb{R}, n$ sufficient large,

(b) $b_n = \sqrt[n]{a^n + b^n}$, $0 \leq a \leq b$, (d) $d_n = \left(\sqrt{n^2+1} - n\right)(n+7)$.

Exercise T11: Compute the limits of the following complex sequences:

(a) $a_n = \dfrac{2+i^n}{4+n}$, $n \in \mathbb{N}$, (b) $b_n = \dfrac{(5n+i)(ni+3)}{(1+i)^n}$, $n \in \mathbb{N}$, (c) $c_n = \dfrac{2(-6)^n i^n + 2^n}{5(-6)^{n+1} i^n + 3^n}$, $n \in \mathbb{N}$.

Exercise T12: Consider the sequence

$$a_n = \frac{1}{2} + (-1)^n \left(1 - \frac{1}{n}\right), \quad n \in \mathbb{N}.$$

Is the sequence bounded? If so, give the smallest possible value for r, for which $|a_n| \leq r$. Justify your answer.

For detailed information regarding this course please check the page
`http://www.mathematik.uni-karlsruhe.de/iag1/lehre/am12009w/en`

Tutorial date: Tuesday, November 17, 2009, 3:45-5:15 pm.