**Karlsruhe Institute of Technology**
Institute for Algebra and Geometry
Dr. T. Arens
Dipl.-Math.techn. A. Schkarbanenko
Dipl.-Math. S. Heumann

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Karlsruhe, November 10, 2009

Student Nr.:

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# Worksheet No.4
# Advanced Mathematics I

**Exercise 16:** Determine all accumulation points of the sequences

$$\text{(a)}\quad a_n = \frac{1}{n} + 2(-1)^n \qquad\qquad \text{(b)}\quad b_n = \left(\frac{5n+7}{n}\right) i^n$$

**Exercise 17:** Examine the following sequences for boundedness, monotonicity and convergence. (Limits or accumulation points need not be specified). If appropriate, analyze suitable subsequences of each sequence.

$$\text{(a)}\quad a_n = \frac{1+6n+2n^2}{(n+3)n}, \qquad \text{(c)}\quad c_n = \frac{(-2)^{-n}+1}{1+2n} - 1 + \frac{2n}{1+2n},$$

$$\text{(b)}\quad b_n = 6 - \frac{6+n^2}{n}, \qquad \text{(d)}\quad d_n = \frac{1+2^n}{1+2^n+(-2)^n}.$$

**Exercise 18:** For $a \in \mathbb{R}_{>0}$ we define the sequence $(x_n)$ by

$$x_{n+1} = 2x_n - ax_n^2, \quad n \in \mathbb{N}_0$$

for arbitrary $0 < x_0 < \frac{1}{a}$.

(a) Show $x_{n+1} \le \frac{1}{a}$ for every $n \in \mathbb{N}_0$ (hint: completion of the square).

(b) Show by induction: $x_n \ge 0$ for every $n \in \mathbb{N}_0$ (hint: use part (a)).

(c) Why does $(x_n)_n$ converge? Determine $\lim\limits_{n\to\infty} x_n$.

**Exercise 19:** We discuss the recursively defined sequence

$$a_1 = b\,, \qquad a_{k+1} = \frac{|a_k|}{2a_k - 1}\,, \; k \in \mathbb{N}$$

for two initial values $b = -\frac{1}{4}$, sowie $b = \frac{1}{4}$.

(a) Compute all possible limits under the assumption of convergence.

(b) Determine for which initial value the sequence is monotonic.

(c) Determine for which initial value the sequence is bounded.

(d) Justify whether the sequence is converging or not for either of the initial values. Determine the limits.

**Exercise 20:** A student memorizes three pages of AM1 lecture notes a day. Overnight he forgets 4% of his total acquired knowledge. Assume that the lecture notes has infinite number of pages and that student has no AM1 knowledge at his first semester day.

(a) Set up the sequence (in a recursive form) for the amount of knowledge $(w_n)_n$ of probant after expiration of $n$ days and $n$ nights.

(b) Prove that $(w_n)_n$ is monotonely increasing.

(c) Prove that $(w_n)_n$ is bounded by 75 pages from above.

(d) What will his level be in the long run?

**Due date:** Please hand in your homework on Thursday, November 26, 12:00, into the AM1-box near Seminar room 1C-03, Allianz-Gebäude (05.20).

# Tutorial 4
# Advanced Mathematics I

**Exercise T13:** Which of the following assertions is true?

(a) A sequence converges, if it is monotonic and bounded.

(b) If a sequence converges, it is monotonic and bounded.

(c) If a sequence is not bounded, it can't be convergent.

(d) If a sequence is not monotonic, it can't be convergent.

(e) If a sequence has exactly one accumulation point, it converges.

(f) If a sequence converges, it has exactly one accumulation point.

**Exercise T14:** Analyze the following sequences to determine whether they are bounded, monotonic, and convergent (in case of convergence you need not specify the limit).

$$(a)\quad a_n = \frac{1+n+n^2}{n(n+1)}\quad,\qquad (b)\quad b_n = \frac{1+n+n^2}{n+1}\quad,$$
$$(c)\quad c_n = \frac{1}{1+(-2)^n}\quad,\qquad (d)\quad d_n = \frac{1+(-2)^n}{1+2^n}\quad.$$

**Exercise T15:** Let the sequence $(a_k)_k$ be defined by

$$a_1 = 1,\quad a_{k+1} = \frac{a_k}{1+\sqrt{1+a_k^2}},\quad k = 1, 2, \dots$$

(a) Show that this sequence converges and compute its limit.

(b) Verify that the sequence $(b_k)_k$ with $b_k = 2^k a_k$, $k = 1, 2, \dots$ converges as well.

**Exercise T16:** Let the sequence $(a_n)$ be recursively defined by the formula

$$a_{n+1} = a_n^2 + \frac{1}{4},\qquad n \in \mathbb{Z}_{\geq 0}.$$

(a) Show that the sequence converges for every initial value $0 \leq a_0 \leq \frac{1}{2}$ and compute its limit.

(b) Show that the sequence diverges for each $a_0 > \frac{1}{2}$ (Hint: Show that $a_n \geq a_0 + nd$ where $d = (a_0 - 1/2)^2$).

(c) What happens when $a_0 < 0$?

For detailed information regarding this course please check the page
`http://www.mathematik.uni-karlsruhe.de/iag1/lehre/am12009w/en`

**Tutorial date:** Tuesday, November 24, 2009, 3:45-5:15 pm.