Karlsruhe Institute of Technology
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21	22	23	24	25	$\sum$

Karlsruhe, November 24, 2009

Student Nr.:

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Worksheet No.5
Advanced Mathematics I

Exercise 21: Let f be a real-valued function defined by

$$f(x) = \frac{x^3 - 3x + 2}{x^3 - 7x + 6}.$$

Determine the maximal domain D and the range $f(D)$ of f.

Exercise 22: Let the polynomial $f(x) = x^4 + 5x^3 - 8x^2 + 1 - (x-2)^3$ be given.

(a) Give the expansion of f around the expansion points $x_1 = 1$ and $x_2 = -1$, as well.

(b) Represent f as a product of linear polynomials. What can you conclude about the behaviour of f in the intervals $[1, \infty)$ and $(-\infty, -3]$?

(c) Sketch the inverse functions of f restricted to $[1, \infty)$ and $(-\infty, -3]$, respectively, and give the appropriate domains and ranges.

Exercise 23: Determine the domain D, the range $f(D)$ and the inverse function f^{-1} of

$$f : \begin{cases} D \longrightarrow \mathbb{R}, \\ x \mapsto 1 - \dfrac{1}{x} \end{cases}.$$

Prepare a sketch of f and f^{-1}.

Exercise 24: Consider the set $K = \{z \in \mathbb{C} \ : \ |z| = 2\}$ and the functions $h\colon \mathbb{C} \to \mathbb{C}$, $z \mapsto \frac{1}{1+z}$ and $g\colon \mathbb{C} \to \mathbb{C}$, $z \mapsto -z$.

(a) What is the geometrical meaning of the map g?

(b) Show that h maps K onto a circle with the centre $-1/3$ and the radius $2/3$.

(c) Now let K be a circle with the centre $-\frac{3}{2} + \frac{i}{2}$ and the radius 2. Determine the range $f(K)$, where

$$f : \begin{cases} \mathbb{C} \setminus \{-5/2 + i/2\} \longrightarrow \mathbb{C}, \\ z \mapsto \dfrac{-2}{2z + 5 - i} \end{cases}.$$

Prepare a sketch of the range $f(K)$ and the inverse image K.

Exercise 25: Let $p(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0$ be a polynomial with real coefficients a_k, $k = 0, \ldots, n-1$ and $z \in \mathbb{C}$. Prove that:

(a) If $p(z) = 0$ then $p(\overline{z}) = 0$.

(b) The product of all zeros of the polynomial p is real.

(c) The sum of all zeros of p is real.

Due date: Please hand in your homework on Thursday, December 3, 12:00, into the AM1-box near Seminar room 1C-03, Allianz-Gebäude (05.20).

Tutorial 5
Advanced Mathematics I

Exercise T17: Let the function $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \frac{1}{8}x^3 + \frac{3}{8}x^2 - \frac{9}{8}x + \frac{5}{8}$.

(a) Give the expansion of f around the expansion points $x_1 = 1$ and $x_2 = 3$, as well. What can you conclude about the behaviour of f in the intervals $[-5, \infty)$ and $(-\infty, 3]$?

(b) With help of a sketch find intervals on which f has an inverse function. Prepare a sketch of the inverse function.

Exercise T18: Let $D \subset \mathbb{R}$. Define the function $f : D \to \mathbb{R}$ by

$$\text{(a) } x \mapsto \frac{x^3 + x^2 - 4x - 4}{x^2 - x - 2}, \qquad \text{(b) } x \mapsto \frac{x^3 - 2x^2 + 3x - 2}{x^2 - 2x + 5}.$$

Specify the maximal domain D of f. Further, for part (a), determine the range $f(D)$ of f and decide if an inverse function $g : f(D) \to D$ exists. Specify it, if it does.

Exercise T19: Let the function $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 - 2x - x^2, & x \leq 1, \\ 9 - 6x + x^2, & x > 1. \end{cases}$$

Determine the largest possible intervals in which the function is invertible. Determine the inverse function in each case and sketch it.

Exercise T20: Given the following subsets of $\mathbb{C}$

$$M_1 := \left\{ \frac{t^2 + 3 + i\,2t}{t^2 + 1} : t \in \mathbb{R} \right\}, \qquad M_2 := \{3 + i\,s : s \in \mathbb{R}\},$$

show that

$$\text{(a)} \quad |z - 2| = 1 \text{ for all } z \in M_1, \qquad \text{(b)} \quad M_2 = \left\{ \frac{2z}{z - 1} : z \in M_1 \right\}.$$

Without proof: Which geometrical figures in the complex plane do M_1 and M_2 represent?

Selftest:

$a_n =$	1	n	$1/n$	$(-1)^n$	$(-1)^n n$	i^n/n	$(-1)^n n + n$	$a_{n-1}/2$, $a_0 = 1$
constant								
bounded								
unbounded								
convergent								
divergent								
improperly convergent								
monotonically decreasing								
strictly decreasing								
monotonically increasing								
strictly increasing								
alternating								
accumulation point(s)								
limit								

Assign the characteristics to different sequences $(a_n)_n$.

Tutorial date: Tuesday, December 1, 2009, 3:45-5:15 pm.