Karlsruhe Institute of Technology
Institute for Algebra and Geometry
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36	37	38	39	40	$\sum$

Karlsruhe, December 15, 2009

Student Nr.:

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Worksheet No.8
Advanced Mathematics I

Exercise 36: Determine the radius of convergence of the following power series:

(a) $\left(\sum_{k=0}^{\infty} \frac{k+2}{2^k} x^k\right)$, (b) $\left(\sum_{k=1}^{\infty} \frac{(2+x)^{2k}}{(2+\frac{1}{k})^k}\right)$, (c) $\left(\sum_{k=0}^{\infty} \frac{3^{k+2}}{2^k} x^k\right)$.

Exercise 37: The functions cosh, sinh satisfy

$$\cosh(x) = \frac{1}{2}(e^x + e^{-x}), \qquad \sinh(x) = \frac{1}{2}(e^x - e^{-x}),\ x \in \mathbb{R}, \qquad \text{with } e^x := \exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

(a) Prove the identity $\cosh(2x) = [\cosh(x)]^2 + [\sinh(x)]^2$ by means of the Cauchy product formula.

(b) Is there a shorter way to prove the identity?

(c) Determine the power series for $f(x) = [\cosh(x)]^2 - [\sinh(x)]^2$.

Exercise 38: Find the power series $\sum_{n=0}^{\infty} a_n\,(x - x_0)^n$ corresponding to the function

$$f(x) = \frac{e^{(x-x_0)}}{1-(x-x_0)}, \qquad x \in \mathbb{R} \setminus \{1 + x_0\}.$$

(a) Show that $a_n = \sum_{k=0}^{n} \frac{1}{k!}$.

(b) For which $x \in \mathbb{R}$ does the power series converge?

Exercise 39: Determine the limit

$$\lim_{x\to 0} \frac{\cos x - 1 + \frac{x^2}{2}}{\exp(x^4) - 1}$$

by substituting the respective power series for cos and exp into the given expression.

Exercise 40: Santa Claus, suffering minor rheumatism, is fed up with all the snow. As an early retirement is out of the question he moves from the north pole to the Maledive Islands and decides to deliver his presents henceforth only to places where the total quantity of snow in the coming years is bounded. His personal weather forecast predicts a snow accumulation in the year $2000 + n$ proportional to

$$a_n(x) := \frac{2^n x^{2n}}{(1+\frac{1}{n})^n},$$

where $x \in \mathbb{R}$ is the distance (in 10,000 km) of a given place from the equator.

(a) Calculate the radius of convergence of the power series $\left(\sum_{n=0}^{\infty} a_n(x)\right)$.

(b) Will Santa Claus continue to deliver presents to Karlsruhe?

Hint: The latitude of Karlsruhe is 49°, the circumference of the earth is 40,000 km.

Due date: Please hand in your homework on Thursday, January 7, 12:00, into the AM1-box near Seminar room 1C-03, Allianz-Gebäude (05.20).

Tutorial 8
Advanced Mathematics I

Exercise T29: Determine the radius of convergence of the power series:

$$\text{(a)} \quad \left(\sum_{k=0}^{\infty} \frac{z^k}{3(k+2)!}\right), \qquad \text{(b)} \quad \left(\sum_{k=1}^{\infty} \frac{z^{2k}\cdot 2^k}{(1+\frac{1}{k})^k}\right), \qquad \text{(c)} \quad \left(\sum_{k=0}^{\infty} k^k z^k\right).$$

Exercise T30:

(a) Compute the power series of the rational function $f : \mathbb{C} \setminus \{1\} \to \mathbb{C}$, where $f(z) = \frac{1+z^2}{1-z}$ at the center of expansion $z_0 = 0$.

(b) Compute the radius of convergence of the series.

(c) For which $z \in \mathbb{C}$ does the power series converge?

Exercise T31: Using the power series

$$\sin(x) = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!} \quad \text{and} \quad \cos(x) = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!}$$

and the Cauchy product, verify the identity $2\sin(x)\cos(x) = \sin(2x)$.

Hint:

$$\sum_{k=0}^{n} \binom{2n+1}{2k+1} = 1 + \sum_{k=1}^{n-1}\left[\binom{2n}{2k} + \binom{2n}{2k+1}\right] + 1 = \sum_{k=0}^{2n} \binom{2n}{k} (1)^k \cdot (1)^{2n-k} \overset{\text{Binomial Theorem}}{=} (1+1)^{2n}.$$

Exercise T32: Determine the limit

$$\lim_{x \to 0} \frac{x^3}{\sin x - x}$$

by substituting in the power series for sin.

Tutorial date: Tuesday, December 22, 2009, 3:45-5:15 pm.