4th Exercise: (a) We guess that $a_k = \dfrac{2^k-(-1)^k}{3}$ where $a_k = a_{k-1} + 2a_{k-2}$ for $k \geq 3$ $(*)$ and $a_1 = a_2 = 1$.

Proof by math. induction:

Initial step: $k=1, k=2$: $a_1 = \dfrac{2^1-(-1)^1}{3} = 1$ ✓ $a_2 = \dfrac{2^2-(-1)^2}{3} = 1$ ✓.

Induction step $k-1, k-2 \to k$: We assume that $(*)$ is valid for $k-1$ and $k-2$, as well. Applying the recursive definition for a_k and the assumption we have

$$a_k = a_{k-1} + 2a_{k-2} \overset{\text{assumption}}{=} \frac{2^{k-1}-(-1)^{k-1}}{3} + 2\,\frac{2^{k-2}-(-1)^{k-2}}{3} =$$

$$= \frac{2^{k-1}+2^{k-1}+(-1)^k-2(-1)^k}{3} = \frac{2^k-(-1)^k}{3}$$

□

(b) Determine the limit $\lim\limits_{n\to\infty} \sqrt[n]{a_n}$ if it exists.

Using the compact/explicit form of a_k we can easyly answer this question:

$$\lim_{n\to\infty} \sqrt[n]{\frac{2^n-(-1)^n}{3}} = \lim_{n\to\infty} \sqrt[n]{\frac{2^n\left(1-\left(\frac{-1}{2}\right)^n\right)}{3}} =$$

$$= \lim_{n\to\infty} \sqrt[n]{2^n} \cdot \sqrt[n]{\frac{1-\left(\frac{-1}{2}\right)^n}{3}} = 2 \cdot \frac{\lim\limits_{n\to\infty} \sqrt[n]{1-\left(\frac{-1}{2}\right)^n}}{\lim\limits_{n\to\infty} \sqrt[n]{3}} \overset{(**)}{=} 2.$$

$(**)$: We claim: $\lim\limits_{n\to\infty} \sqrt[n]{1-\left(\frac{-1}{2}\right)^n} = 1.$

Proof by sandwich theorem:

$$\sqrt[n]{1-\tfrac{1}{2}} \leq \sqrt[n]{1-\left(\tfrac{-1}{2}\right)^n} \leq \sqrt[n]{1+1}, \quad n \in \mathbb{N}$$

(by ex. 2.4) $(n\to\infty)$ $n\to\infty$ (by example 2.4)

$\boxed{1}$

□