3rd Exercise: Consider the sets $G_1 := \{ z \in \mathbb{C} : |z-3| = |z-3-4i| \}$, $G_2 := \{ z \in \mathbb{C} : |z| = |z-i| \}$ and the functions $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$, $g : \mathbb{C} \setminus \{-\frac{3}{2}i\} \to \mathbb{C}$ given by

$$f(z) = \frac{1}{z} \quad \text{and} \quad g(z) = \frac{1}{z + \frac{3}{2}i} \quad \text{respectively.}$$

(b) Show that the range $f(G_1)$ is included in a circle with centre $-\frac{1}{4}i$ and radius $\frac{1}{4}$, i.e. $f(G_1) \subseteq \{ z \in \mathbb{C} : |z + \frac{1}{4}i| = \frac{1}{4} \} =: K$.

(c) Show that the range $g(G_2)$ is included in K, as well!

Solution: (b) To show that $z \in G_1$ fulfills the condition

$$|f(z) + \tfrac{1}{4}i| = \tfrac{1}{4} \qquad \left(\Leftrightarrow |f(z) + \tfrac{1}{4}i|^2 = \left(\tfrac{1}{4}\right)^2 \right). \qquad (*)$$

Let $z \in G_1 \Rightarrow \operatorname{Im}(z) = 2$, i.e. $z = x + 2i$, $x \in \mathbb{R}$. Insert this into $(*)$.

$$\left| \frac{1}{x+2i} + \frac{1}{4}i \right|^2 = \left| \frac{x-2i}{x^2+4} + \frac{1}{4}i \right|^2 \overset{|z|^2 = z \cdot \bar{z}}{=} \frac{|x-2i|^2}{(x^2+4)^2} + \frac{1}{4}i \cdot \frac{x+2i}{x^2+4} -$$

$$- \frac{1}{4}i \, \frac{x-2i}{x^2+4} + \frac{1}{16} =$$

$$= \frac{x^2+4}{(x^2+4)^2} + \frac{1}{4}i \, \frac{x}{x^2+4} - \frac{1}{4}i \, \frac{x}{x^2+4} - \frac{1}{4} \, \frac{2}{x^2+4} - \frac{1}{4} \cdot \frac{2}{x^2+4} + \frac{1}{16}$$

$$= \frac{1}{x^2+4} - \frac{1}{x^2+4} + \frac{1}{16} = \frac{1}{16} \quad \checkmark$$

Thus $f(G_1) \subseteq K$.

(c) There is a simple relationship between f and g:

$$g(z) = \frac{1}{z + \frac{3}{2}i} =: \frac{1}{h(z)} = f(h(z))$$

$h(z)$: shifting z by $\frac{3}{2}i$

Thus $g(G_2) = f(h(G_2)) = f(G_1) \subseteq K$ by part (b).

see the sketch
