1st Exercise: Determine for $x \in \mathbb{R}$ the domain of convergence for

$$f(x) = \sum_{k=0}^{\infty} \frac{4^k (2k+3)}{k^2+3k+2} (x-3)^{2k}.$$

Solution: ∘ Apply ratio test:

$$\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = \lim_{k\to\infty} \left| \frac{4^{k+1}(2(k+1)+3)(x-3)^{2k+2}}{(k+1)^2+3(k+1)+2} \right| \cdot \left| \frac{k^2+3k+2}{2^k(2k+3)(x-3)^{2k}} \right| =$$

$$= 4 \cdot \lim_{k\to\infty} \underbrace{\frac{|2k^3+11k^2+19k+10|}{|2k^3+13k+27k+18|}}_{\to 1\ (k\to\infty)} \cdot |x-3|^2 = 4 \cdot |x-3|^2 \overset{!}{<} 1$$

So, $|x-3|^2 \overset{!}{<} \frac{1}{4} \Rightarrow |x-3| < \frac{1}{2} \Rightarrow x \in (3-\frac{1}{2}, 3+\frac{1}{2})$.

Root test yields the same information.

- Now consider $x = 3-\frac{1}{2}$, $x = 3+\frac{1}{2}$.

$x = 3+\frac{1}{2}$: $0 < \sum_{k=0}^{\infty} \frac{4^k(2k+3)}{k^2+3k+2}\left(\frac{1}{2}\right)^{2k} = \sum_{k=0}^{\infty} \frac{2k+3}{k^2+3k+2} \geq \frac{3}{2}\ 5 \underbrace{\sum_{k=1}^{\infty} \frac{1}{k}}_{\text{harm. series!}}$

$\Rightarrow$ Divergence by comparison test.

$x = 3-\frac{1}{2}$: $0 < \sum_{k=0}^{\infty} \frac{4^k(2k+3)}{k^2+3k+2}\left(-\frac{1}{2}\right)^{2k} = \sum_{k=0}^{\infty} \frac{2k+3}{k^2+3k+2}(-1)^{2k}$.

$\Rightarrow$ Divergence by comparison test!

- Thus the domain of convergence is $(3-\frac{1}{2}, 3+\frac{1}{2})$.