Karlsruhe Institute of Technology
Institute for Algebra and Geometry
Dr. habil. G. Fels
Dipl.-Math.techn. A. Schkarbanenko
Dipl.-Math.techn. S. Schmitt

51	52	53	54	55	$\sum$

Karlsruhe, January 18, 2011

Student Nr.:

....

....

Worksheet No.11
Advanced Mathematics I

Exercise 51: Use the Mean Value Theorem to prove the following inequalities:

$$\text{(a) } |\cos e^x - \cos e^y| \le |x-y| \text{ für } x, y \le 0\,, \qquad \text{(b) } \ln(1+x) \le \frac{x}{\sqrt{1+x}} \text{ für } x > 0\,.$$

Hint for (b): Consider $f(t) = \ln(1+t) - \frac{t}{\sqrt{1+t}}$ on the interval $[0, x]$.

Exercise 52: Compute the following limits:

$$\text{(a) } \lim_{x\to 0} \frac{\tan x - x}{x^3}, \quad \text{(b) } \lim_{x\to\infty} \left(x + \frac{1}{\ln\left(1-\frac{1}{x}\right)} \right), \quad \text{(c) } \lim_{x\to\frac{\pi}{2}} \frac{\ln(\frac{\pi}{2}-x)}{\tan x}, \quad \text{(d) } \lim_{x\to 0} x^{\tan x}, \quad \text{(e) } \lim_{x\to\infty} \frac{\sinh x}{\cosh x}\,.$$

Hint: For (b) first compute $\left(\ln\left(1-\frac{1}{x}\right)\right)'$.

Exercise 53: Compute the following limits:

$$\text{(i)} \quad \lim_{x\to\infty} \frac{x^2\, e^x}{(e^x-1)^2}, \qquad \text{(ii)} \quad \lim_{x\to 0} \frac{\sin^2 x}{\sin(x^2)}.$$

and determine the constant $c \in \mathbb{R}$, so that the function

$$f(x) = \begin{cases} c, & x = 1, \\ \frac{2^{\ln x} - x}{\ln x}, & x \ne 1, \end{cases}$$

is continuous in $\mathbb{R}_{>0}$.

Exercise 54: Consider the function $f(x) = \sqrt{x+1},\ x \ge 0\,.$

(a) Prove that the derivatives of f are

$$f^{(k)}(x) = -(-1)^k \frac{(2k-2)!}{2^{2k-1}\,(k-1)!}(1+x)^{-\frac{2k-1}{2}}\,.$$

(b) Compute the taylor series of f with respect to the expansion point $x_0 = 0$.

(c) Determine whether the series

$$\left(\sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{4^n (n!)^2 (2n-1)} \right)$$

is convergent, and if so compute its value.

Exercise 55: Use Taylor's formula of second degree with expansion point $x_0 = 0$ and the remainder term in Lagrange form to prove the estimate

$$x - \frac{1}{2}x^2 + \frac{1}{3}\left(\frac{x}{1+x}\right)^3 \le \ln(1+x) \le x - \frac{1}{2}x^2 + \frac{1}{3}x^3, \qquad 0 \le x < \infty.$$

Due date: Please hand in your homework until Thursday, January 27, 12:00 into the AM1/2-box near seminar room Z1, building 01.85 (Fritz-Erler-Str. 1-3).

Tutorial 11
Advanced Mathematics 1

Exercise T31: Compute the limits

(a) $\lim\limits_{x\to 0} \dfrac{\sqrt{a+2x}-\sqrt{a+x}}{x}, \quad a>0,$ (b) $\lim\limits_{x\to 0} \dfrac{1}{x}\left(\cot x - \dfrac{1}{x}\right),$

(c) $\lim\limits_{x\to 0} \left(\cot(x)\arcsin(x)\right),$ (d) $\lim\limits_{x\to 0+} \dfrac{\ln\tan 7x}{\ln\tan 2x}.$

Exercise T32: Compute the Taylor polynomial of degree 2 with center $x_0 = 0$ for the function $f(x) = \sqrt[3]{1+x}$, $-1 < x$ and estimate the error.

Exercise T33: Compute all the derivatives $f^{(n)}$, $n = 0, 1, 2, \ldots$ of the function f and use these to construct the Taylor series of f having center $x_0 = 0$.

(a) $f(x) = \cos\dfrac{x}{2},\ x \in \mathbb{R},$ (b) $f(x) = \ln\dfrac{2+x}{2-x},\ |x| < 2.$

Where do the resulting series converge?

Please note the following deadlines with respect to the exam:

- It is possible to register starting from 31 January 2011.
- The online registration has to be carried out by 15 February 2011. For later cancellations, please see the staff at the institute.

For detailed information please check the page
`www.math.kit.edu/iag1/lehre/hm1mach2010w/event/klausurhm12011w/en`

Tutorial date: Wednesday, January 19, 2010, 14:00-15:30.