Aufgabe 6: a) $\int x^2 \sin x dx = x^2(-\cos x) + \int 2x \cos x dx = -x^2 \cos x + 2\left[x \sin x - \int \sin x dx\right]$
$= -x^2 \cos x + 2x \sin x + 2 \cos x + C.$
b) $\int \arctan \frac{1}{x-1} dx = x \arctan \frac{1}{x-1} - \int x \frac{1}{1+\left(\frac{1}{x-1}\right)^2} \frac{-1}{(x-1)^2} dx = x \arctan \frac{1}{x-1} + \int \frac{x}{(x-1)^2+1} dx$
$= x \arctan \frac{1}{x-1} + \frac{1}{2} \int \frac{2(x-1)}{(x-1)^2+1} dx + \int \frac{dx}{(x-1)^2+1} = x \arctan \frac{1}{x-1} + \frac{1}{2} \ln\left[(x-1)^2+1\right] + \arctan(x-1) + C.$
c) Wegen $\int \ln y dy = y \ln y - y + C$ ist $\int (\ln y)^2 dy = (y \ln y - y) \ln y - \int (y \ln y - y) \frac{1}{y} dy = y(\ln y - 1) \ln y$
$- \int (\ln y - 1) dy = y(\ln y - 1) \ln y - [y \ln y - y - y] + C = y \ln^2 y - 2y \ln y + 2y + C = y(\ln y - 1)^2 + y + C.$
d) $\int x^3 e^x dx = [x^3 e^x] - \int 3x^2 e^x dx = x^3 e^x - ([3x^2 e^x] - \int 6x e^x dx) = x^3 e^x - 3x^2 e^x + [6x e^x] - \int 6 e^x dx =$
$x^3 e^x - 3x^2 e^x + 6x e^x - 6 e^x.$

Aufgabe 7: a) Substitution $x = u^2$, $dx = 2u du$ liefert $\int \frac{(1+u^2)^2}{u} 2u du = 2 \int (1 + 2u^2 + u^4) du =$
$2u + \frac{4}{3} u^3 + \frac{2}{5} u^5.$
Resubstitution liefert $\int \frac{(1+x)^2}{\sqrt{x}} dx = 2\sqrt{x} + \frac{4}{3} x^{\frac{3}{2}} + \frac{2}{5} x^{\frac{5}{2}} + C$ für $x > 0$.
b) $t = \sqrt{y-1}$ für $t > 0$, $dt = \frac{dy}{2\sqrt{y-1}}$ liefert $\int \frac{(1+y)(y-1)^{\frac{3}{2}}}{y^3} \frac{dy}{2\sqrt{y-1}} = \frac{1}{2} \int \frac{y^2-1}{y^3} dy = \frac{1}{2} \int \left(\frac{1}{y} - \frac{1}{y^3}\right) dy$
$= \frac{1}{2}\left(\ln|y| - \frac{1}{-2y^2}\right)$. Resubstitution $\int \frac{(2+t^2)t^3}{(1+t^2)^3} dt = \frac{1}{2}\left(\ln(1+t^2) + \frac{1}{2} \frac{1}{(1+t^2)^2}\right) + C$ für $t > 0$. Man rechnet leicht nach, daß das sogar für alle $t \in \mathbb{R}$ gilt.
c) $\int_0^a a^r dr = \int_0^{a \ln a} a^{\frac{w}{\ln a}} \frac{dw}{\ln a} = \frac{1}{\ln a} \int_0^{a \ln a} e^{\frac{w}{\ln a} \ln a} dw = \frac{1}{\ln a} \left[e^w\right]_0^{a \ln a} = \frac{1}{\ln a}(a^a - 1).$
d) $w = (v-1)^2$, $dw = 2(v-1)dv \Rightarrow \int \frac{v^3}{v-1} 2(v-1) dv = 2\frac{v^4}{4}. \Rightarrow \int \frac{(1+\sqrt{w})^3}{\sqrt{w}} dw = \frac{1}{2}(1+\sqrt{w})^4 + C$, $w > 0$.

Aufgabe 8: a) $\int_{-5}^{-3} x^{2n+1} dx = \left[\frac{x^{2n+2}}{2n+2}\right]_{-5}^{-3} = \frac{(-3)^{2(n+1)} - (-5)^{2(n+1)}}{2(n+1)} = \frac{9^{n+1} - 25^{n+1}}{2(n+1)} < 0.$
b) $\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^2}} = [\arcsin x]_{-\frac{1}{2}}^{\frac{1}{2}} = \arcsin \frac{1}{2} - \arcsin \frac{-1}{2} = \frac{\pi}{6} - \frac{-\pi}{6} = \frac{\pi}{3}.$
c) Substitution: $x(t) = \frac{1}{2}(t^2 - 1)$ auf $[1, \sqrt{3}]$ liefert $\int_0^1 3^{\sqrt{2x+1}} dx = \int_1^{\sqrt{3}} 3^t t dt \overset{part.\ integr.}{=}$
$= \left[\frac{3^t}{\ln 3} t\right]_1^{\sqrt{3}} - \int_1^{\sqrt{3}} \frac{3^t}{\ln 3} dt = \left[\frac{3^t}{\ln 3} t\right]_1^{\sqrt{3}} - \frac{1}{\ln 3}\left[\frac{3^t}{\ln 3}\right]_1^{\sqrt{3}} = \frac{1}{\ln 3}\left[3^{\sqrt{3}}\sqrt{3} - 3 - \frac{3^{\sqrt{3}}}{\ln 3} + \frac{3}{\ln 3}\right].$
d) $\int_1^2 \frac{(\ln x)^3}{x^2} dx = \left[\frac{-1}{x}(\ln x)^3\right]_1^2 - \int_1^2 \frac{-1}{x} 3(\ln x)^2 \frac{1}{x} dx = \frac{-1}{2}(\ln 2)^3 + 3 \int_1^2 \frac{(\ln x)^2}{x^2} dx$
$= \frac{-1}{2}(\ln 2)^3 + 3\left(\left[\frac{-1}{x}(\ln x)^2\right]_1^2 - \int_1^2 \frac{-1}{x} 2 \ln x \frac{1}{x} dx\right) = \frac{-1}{2}(\ln 2)^3 - \frac{3}{2}(\ln 2)^2 + 6 \int_1^2 \frac{\ln x}{x^2} dx$
$= \frac{-1}{2}(\ln 2)^3 - \frac{3}{2}(\ln 2)^2 + 6\left(\left[\frac{-1}{x} \ln x\right]_1^2 - \int_1^2 \frac{-1}{x} \frac{1}{x} dx\right) = \frac{-1}{2}(\ln 2)^3 - \frac{3}{2}(\ln 2)^2 - 3 \ln 2 + 3.$

Aufgabe 9: $\int_0^x \sin^n u du = \int_0^x \sin u \sin^{n-1} u du = \left[-\cos u \sin^{n-1} u\right]_0^x + \int_0^x \cos u (n-1) \sin^{n-2} u \cos u du =$
$-\cos x \sin^{n-1} x$
$+ (n-1) \int_0^x (1 - sin^2 u) sin^{n-2} u du \Rightarrow n \int_0^x \sin^n u du = -\cos x \sin^{n-1} x + (n-1) \int_0^x \sin^{n-2} u du. \quad (*)$
Beh.: $\int_0^{\pi/2} \sin^{2m} u du = \frac{2m-1}{2m} \cdot \frac{2m-3}{2m-2} \cdot \ldots \cdot \frac{1}{2} \frac{\pi}{2} \quad (m \in \mathbb{N}) \quad (**)$
Bew.: vollständige Induktion
Induktionsanfang: $m = 1$: $\int_0^{\pi/2} \sin^2 u du = \int_0^{\pi/2} \frac{1 - \cos 2u}{2} du = \left[\frac{u}{2} - \frac{\sin 2u}{4}\right]_0^{\pi/2} = \frac{\pi}{4} = \frac{1}{2} \frac{\pi}{2}.$
Induktionsvoraussetzung: $(**)$ gilt für irgendein $m \in \mathbb{N}$.
Ind.-Schritt: $\int_0^{\pi/2} \sin^{2(m+1)} u du \overset{(*)}{=} -\frac{1}{n} \sin^{2m+1} \frac{\pi}{2} \cos \frac{\pi}{2} + \frac{2m+1}{2m+2} \int_0^{\pi/2} \sin^{2m} u du \overset{(**)}{=} \frac{2m+1}{2m+2} \cdot \frac{2m-1}{2m} \cdot \frac{2m-3}{2m-2} \cdot \ldots \cdot \frac{1}{2} \frac{\pi}{2}.$
Die andere Formel zeigt man analog.

Aufgabe 10: a) $y = \int \frac{1}{1+x^2} = \arctan x + C$
b) Getrennte Veränderliche: $\frac{dy}{dx} = x^2 y \Leftrightarrow \frac{dy}{y} = x^2 dx \Rightarrow \int \frac{dy}{y} = \int x^2 dx \Rightarrow \ln(y) = x^3/3 + C \Rightarrow y =$
$e^{x^3/3+C} = D \cdot e^{x^3/3}$
c) Getrennte Veränderliche: $y^2 dy = \frac{x}{\sqrt{1+x^2}} dx \Rightarrow \int y^2 dy = \int \frac{x}{\sqrt{1+x^2}} dx \Rightarrow \frac{y^3}{3} = \sqrt{1+x^2} + C$ ist die

Lösung in impliziter Form.