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Karlsruhe, 16.06.2008

Student Nr.: ....

Worksheet No.11
Advanced Mathematics II

Exercise 51: (a) Compute the gradient ∇f and the differential df of the function

$$f(x_1, x_2, x_3) = \sin(x_1)x_2\cos(x_3).$$

(b) Compute the Jacobian matrix f' for the function

$$f(x_1, x_2, x_3) \quad = \begin{pmatrix} \sin(x_1) + \ln(1 - x_2^2) \\ e^{-x_2^2} \\ \cosh(x_1 x_3) \end{pmatrix}$$

and determine the domain of f.

Exercise 52: A function $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = (x^2 - 2y^2)e^{x^2+y^2}$.

(a) Find the gradient of the function $g(t) = f(\sin(t), \cos(t))$, employing the chain rule of differentiation, and determine all critical points of g.

(b) Moreover, determine a point $x_0 \in \mathbb{R}^2$ and a direction a such that the directional derivative of f along a vanishes at x_0.

Exercise 53: Consider the function $f : \mathbb{R}^3 \to \mathbb{R}$ with

$$f(x_1, x_2, x_3) = x_1 \cos x_2 \cos x_3, \quad x_1, x_2, x_3 \in \mathbb{R}.$$

Determine the gradient ∇f and the Hessian matrix $H_f(x) = \left(\dfrac{\partial^2 f(x)}{\partial x_i \partial x_j}\right)_{i,j=1,2,3}$.

Exercise 54:

(a) Determine the gradient of the scalar function $f : (-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \int_0^{x_1} \tan(t)\, \exp(x_2 \cos t)\, dt\,.$$

(b) Calculate the derivative of the function $g : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ defined by $g(s) = f(s, \cos s)$ at $s = \pi/4$.

Exercise 55: Consider the function

$$f(x, y) = \arctan\left(\frac{y}{x}\right), \quad x \in \mathbb{R} \setminus \{0\}, \quad y \in \mathbb{R}.$$

(a) Determine df in the point $p_0 = \left(1, \frac{1}{\sqrt{3}}\right)$ and obtain the equation for the tangential plane in p_0.

(b) In which direction $(dx, dy) = (\cos\alpha, \sin\alpha)$ does the tangential plane in p_0 have zero slope? What are the directions of the strongest increase, respectively, the strongest decrease of $f(x, y)$ in p_0?

Due date: Please hand in your homework on Wednesday, July 2, 09:30.

Tutorial 10
Advanced Mathematics II

Ecercise T37:

(a) Consider vector valued functions f and g defined as follows

$$f(x,y) = \begin{pmatrix} xy \\ x^2+y^2 \end{pmatrix}, \quad g(x,y) = \begin{pmatrix} \cos x \\ \sin y \end{pmatrix}, \quad x,y \in \mathbb{R}.$$

Calculate $(f^\top g)'$ and $(f \circ g)'$.

(b) Determine for the vector valued functions

$$f(x,y) = \begin{pmatrix} xy \\ x^2-y^2 \end{pmatrix} \quad \text{und} \quad g(x,y) = \begin{pmatrix} y\cosh x \\ y\ \sinh x \end{pmatrix}, \quad x,y \in \mathbb{R},$$

the derivative $(f \circ g)'$ directly and by chain rule as well.

Ecercise T38:

(a) Calculate the derivative (Jacobian) of the functions

$$f : \begin{cases} \mathbb{R}^3 \to \mathbb{R}^2 \\ x = (x_1,x_2,x_3)^\top \mapsto \dfrac{(x_2,x_3)^\top}{1+(x_1+x_2+x_3)^2} \end{cases}, \qquad g : \begin{cases} \mathbb{R}^2 \to \mathbb{R} \\ x = (x_1,x_2)^\top \mapsto \cos(x_1x_2) \end{cases}.$$

(b) Define matrices $A := f'(1,0,1)$, $B := \nabla g(\pi,1)$, $C := g'(1,\pi)$. Which of the following matrix products are defined: AB, BA, AC, CA ?

Ecercise T39: Consider the scalar function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x) := x_1^2x_2$, and a vector $d := (\cos\varphi, \sin\varphi)^\top$, $\varphi \in [0,2\pi)$.

(a) Determine the gradient ∇f and the scalar product $d \cdot \nabla f$ at x.

(b) Calculate the directional derivative $\dfrac{\partial f}{\partial d}(x)$ using the Definition 3.40.

Ecercise T40: Consider a function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) = \begin{cases} \dfrac{x_1^3x_2 - x_1x_2^3}{x_1^2+x_2^2}, & x \neq (0,0)^\top, \\ 0, & x = (0,0)^\top \end{cases}.$$

Using the definition 3.40 determine the partial derivatives $\dfrac{\partial f}{\partial x_1}(0,t)$ and $\dfrac{\partial f}{\partial x_2}(t,0)$ for all $t \in \mathbb{R}$. In addition, determine $\dfrac{\partial^2 f}{\partial x_1 \partial x_2}(0,0)$ as well as $\dfrac{\partial^2 f}{\partial x_2 \partial x_1}(0,0)$. What can you conclude for f by the Schwarz Theorem?

Tutorial date: Wednesday, June 25.