**Exercise** Consider $f(t) = \int_{t/2}^{t^2} 2\tau t^2 \, d\tau$.

(a) Compute first the integral and then $f'(t)$.

(b) ~~Compute $f'(t)$ by the chain rule.~~
Show that $f(t) = (h \circ \phi)(t)$,

$$h(t,y) = \int_{t/2}^{t^2} 2\tau y \, d\tau, \quad \phi(t) = (t, t^2)^T,$$

and compute $f'(t)$ by the chain rule.

**Solution**

(a)
$$f(t) = \int_{t/2}^{t^2} 2\tau t^2 d\tau$$
$$= 2t^2 \int_{t/2}^{t^2} \tau \, d\tau = 2t^2 \left[ \tfrac{1}{2} \tau^2 \right]_{t/2}^{t^2}$$
$$= 2t^2 \left[ \tfrac{1}{2} t^4 - \tfrac{1}{2} \tfrac{t^2}{4} \right] = t^6 - \frac{t^4}{4}$$
$$f'(t) = 6t^5 - t^3$$

(b)
$$(h \circ \phi)(t) = h(\phi(t)) = h(t, t^2)$$
$$= \int_{t/2}^{t^2} 2\tau t^2 \, d\tau = f(t).$$

Chain rule: $f'(t) = h'(\phi(t)) \cdot \phi'(t)$

Matrix product!

$$\phi'(t) = \begin{pmatrix} 1 \\ 2t \end{pmatrix}$$

$h'(t) = ?$ $\quad h'(t) = \left( \frac{\partial h}{\partial x}, \frac{\partial h}{\partial y} \right)$

Remark: $a(t) = \int_{u(t)}^{v(t)} b(\tau) d\tau$. $\quad a'(t) = ?$ $\quad$ Let $B'(\tau) = b(\tau)$

$$\Rightarrow \quad a(t) = B(v(t)) - B(u(t))$$
$$a'(t) = B'(v(t))\, v'(t) - B'(u(t))\, u'(t)$$
$$= b(v(t))\, v'(t) - b(u(t))\, u'(t)$$

Hence:
$$\frac{\partial h}{\partial x} = \frac{\partial}{\partial x}\left( 2y \int_{x/2}^{x^2} \tau \, d\tau \right) = 2y \frac{\partial}{\partial x}\left( \int_{x/2}^{x^2} \tau \, d\tau \right)$$
$$= 2y \left( x^2 \cdot 2x - \frac{x}{2} \cdot \frac{1}{2} \right)$$
$$= 4x^3 y - \frac{xy}{2}$$

$$\frac{\partial h}{\partial y} = \frac{\partial}{\partial y}\left(\int_{x/2}^{x^2} 2\tau y \, d\tau\right)$$

$$\overset{\text{Th. 4.3}}{=} \int_{x/2}^{x^2} 2\tau \frac{\partial}{\partial y}(y)\, d\tau$$

$$= 2\int_{x/2}^{x^2} 2\tau \, d\tau$$

$$= 2\left(\frac{1}{2}x^4 - \frac{1}{8}x^2\right) = x^4 - \frac{1}{4}x^2$$

Hence: $f'(t) = \left(\frac{\partial h}{\partial x}(t, t^2), \frac{\partial h}{\partial y}(t, t^2)\right) \cdot \begin{pmatrix} \phi_1'(t) \\ \phi_2'(t) \end{pmatrix}$

$$= \left(4t^3 t^2 - \frac{t t^2}{2}, t^4 - \frac{1}{4}t^2\right) \cdot \begin{pmatrix} 1 \\ 2t \end{pmatrix}$$

$$= 4t^5 - \frac{t^3}{2} + 2t^5 - \frac{1}{2}t^3$$

$$= 6t^5 - t^3$$

That's just the result from (a)!