Universität Karlsruhe (TH)
Institut für Algebra und Geometrie
PD Dr. F. Hettlich
Dipl.-Math. techn. A. Schkarbanenko

46	47	48	49	50	Σ

Karlsruhe, 23.6.2009

Student Nr.: ....

Worksheet No. 10
Advanced Mathematics II

Exercise 46: Determine the Laplace-transform of:

(a) $f(x) = x^2 + 3x + 4 + x^2 \sin(2x)$ (b) $f(x) = \begin{cases} \sin(x) & 0 \le x < \pi \\ \cos(x) & x \ge \pi \end{cases}$

(c) $f(x) = (e^{2x} + e^{3x}) \cdot \sin(4x)$ (d) $f(x) = \cos(x) - x\sin(x) = (x \cdot \cos(x))'$

(e) $f(x) = x^n$, $n \in \mathbb{N}$

Use in part (e) the definition of the Laplace-transform and apply mathematical induction.

Exercise 47: Specify which functions $f(t)$, $t \in [0, \infty)$ have been Laplace-transformed in the given examples:

(a) $F(s) = \dfrac{2s}{s^4 + 2s^3 + 2s^2 + 2s + 1}$, (b) $F(s) = \operatorname{arccot}(s-1)$, (c) $F(s) = \dfrac{e^{-\pi s}}{\sqrt{s^2+1}}$.

Hint: Apply in (a) the partial fraction decomposition. In parts (b) and (c) adopt appropriate computational rules of the Laplace-transform.

Exercise 48: Solve the initial value problem

$$y'''(x) + 4y''(x) + 5y'(x) + 2y(x) = x, \quad y(0) = 0,\ y'(0) = 1,\ y''(0) = -1$$

by means of a Laplace-transform.

Exercise 49: Determine the solution of the initial value problem

$$y'(x) = \begin{pmatrix} 4 & 4 \\ -2 & 0 \end{pmatrix} y(x), \quad y(0) = \begin{pmatrix} 3 \\ -1 \end{pmatrix}, \quad x \ge 0,$$

by means of a Laplace-transform.

Exercise 50: Consider the vector space $C[0,1]$ with its scalar product $\langle \cdot, \cdot \rangle$ as defined in exercise No.10 on 2^{nd} worksheet. The functions $v_j(x) = \sin(\pi j x)$, $j = 1, \ldots, n$ span a subspace U of $C[0,1]$. We have $v_j \perp v_k$ for $j \ne k$. Let $u = \sum_{j=1}^n a_j v_j \in U$ (where $a_j \in \mathbb{R}$) be a function which satisfies $\langle u'' - u, v \rangle = \langle f, v \rangle$ with $f(x) = x$ for all $v \in U$.

(a) Show that $\langle u'' - u, v_k \rangle = -\frac{1}{2} a_k (\pi^2 k^2 + 1)$ for $k = 1, \ldots, n$.

(b) Verify that $\langle f, v_k \rangle = \dfrac{(-1)^{k+1}}{\pi k}$ for $k = 1, \ldots, n$.

(c) Determine the coefficients a_k and find the approximative solution u for $n = 3$.

Note: For the case $U = C[0,1]$ we obtain an exact solution of the differential equation $u''(x) - u(x) = f(x)$. The restriction on a subset $U \subsetneqq C[0,1]$ yields an approximative solution for the initial value problem $u(0) = u(1) = 0$. In mathematics this approach is called *Galerkin method*. It is applied particularly for the *finite element method*.

Due date: Please hand in your homework on Thursday, July 2, 11:15 a.m.

Tutorial No. 10
Advanced Mathematics II

Exercise T37: Determine the Laplace-transforms of the following real-valued functions:

(a) $f(t) = \cosh(t)$ (b) $f(t) = t\cosh(t)$ (b) $f(t) = \sinh(\omega t)$ (d) $f(t) = \cos^2(\omega t)$, $\omega > 0$.

Exercise T38: Specify which functions $f(t)$ have been Laplace-transformed in the given examples:

(a) $F(s) = \dfrac{1}{s^5}$ (b) $F(s) = \dfrac{6}{(s-2)^4}$ (c) $F(s) = \dfrac{1}{s(s+1)^2}$ (d) $F(s) = \dfrac{2s}{(s+1)^2(s^2+1)}$.

Exercise T39:

(a) Determine $\mathcal{L}((3-t^2)\sin t - 3t\cos t)(s)$.

(b) Solve the initial value problem

$$u^{(4)}(t) + 2u''(t) + u(t) = \sin t, \quad u(0) = u'(0) = u''(0) = u'''(0) = 0.$$

Exercise T40: Solve the initial value problem by means of the Laplace-transform:

$$\begin{aligned} x'(t) &= 2y(t) + 1 \\ y'(t) &= -2x(t) + 2t \end{aligned}$$

$$x(0) = 0,\ y(0) = 1$$

Tutorial date: Monday, June 29, 3:45 p.m.–5:15 p.m.