Universität Karlsruhe (TH)
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Karlsruhe, 7.7.2009

Student Nr.: ....

Worksheet No. 12
Advanced Mathematics II

Exercise 56: Consider the function $f : \mathbb{R}^3 \to \mathbb{R}$ with

$$f(x_1, x_2, x_3) = x_1 \cos(x_2)\cos(x_3), \quad x_1, x_2, x_3 \in \mathbb{R}.$$

Determine the gradient $\nabla f(x)$ and the Hessian matrix $H_f(x) = \left(\frac{\partial^2 f(x)}{\partial x_i \partial x_j}\right)_{i,j=1,2,3}$ (i.e. the matrix whose elements are the second partial derivatives of f).

Exercise 57: Consider the scalar function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x) := x_1^2 x_2$, and a vector $d := (\cos\varphi, \sin\varphi)^\top$, $\varphi \in [0, 2\pi)$.

(a) Determine the gradient ∇f and the scalar product $d \cdot \nabla f$ at x.

(b) Calculate the directional derivative $\frac{\partial f}{\partial d}(x)$ using Definition 4.14.

Exercise 58:

(a) Determine $(f^T g)'$ and $(f \circ g)'$ for

$$f(x_1, x_2) = \begin{pmatrix} x_1 + x_2^2 \\ 2x_2^3 \end{pmatrix}, \quad g(x_1, x_2) = \begin{pmatrix} x_2 \sin(x_1) \\ \cos(x_1) + x_2 \end{pmatrix}, \quad x_1, x_2 \in \mathbb{R}.$$

(b) Calculate $(f \circ g)'$ directly and then as well by means of the chain rule for

$$f(x_1, x_2) = \begin{pmatrix} x_1 + 1 \\ x_2^2 \end{pmatrix} \quad \text{and} \quad g(x_1, x_2) = \begin{pmatrix} x_1 \cosh(x_2) \\ x_1 \end{pmatrix}.$$

Exercise 59: Find the derivative of the function $f : \mathbb{R}_{>0} \to \mathbb{R}$ given by

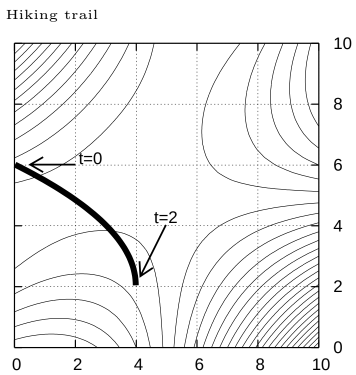
$$f(s) := \int_{1/s}^{s^2} \frac{\sin(st)}{t}\, dt\,,$$

applying partial derivatives of $g(x, y, z) = \int_x^y \frac{\sin(zt)}{t}\, dt$ and the chain rule.

Exercise 60: Mr K. hikes around in the black forest, the topography of which can be described by the scalar function

$$f(x, y) = x^2 y - xy^2 + 3xy - 5x^2 + 10x + 5y^2 - 40y + 500,$$

where $x, y \in [0, 10]$ are coordinates (given in kilometers) and $f(x, y)$ indicates the altitude. During the first two hours, $t \in [0, 2]$, his way runs along the function $c(t) = (x, y)^\top = (4t - t^2, 6 - 2t)^\top$.



(a) Find the gradient of f.

(b) Show that $x = 5$ and $y = 5$ are two contour lines by verifying that f is constant there. Show also that the gradient is orthogonal to these contour lines.

(c) Determine the hiking velocity $\|c'(t)\|$ and hiking direction $r(t) = \frac{1}{\|c'(t)\|} c'(t)$ of Mr K., at time t and the slope of the way for $t = 1$, i.e. the directional derivative of f with respect to the hiking direction on $c(1)$.

(d) Give the derivative of function $f \circ c$ by means of chain rule. What is the vertical velocity of Mr K. at $t = 1$?

Due date: Please hand in your homework on Thursday, July 16, 11:15 a.m.

Tutorial No. 12
Advanced Mathematics II

Exercise T45:

(a) Find the derivative (i.e. the Jacobian) $f'(x)$ and gradient ∇f for the function

$$f(x_1, x_2, x_3) := \frac{x_1}{\cos(x_1 + x_2 + 2x_3)}.$$

(b) What is the derivative of function given by

$$g(x_1, x_2, x_3) := \frac{(x_1, x_2, x_3)^\top}{1 + x_1 + x_2 + x_3}\ ?$$

Exercise T46: Given are the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x) := 3x_1 + x_2^2$ and a vector $d := (\cos\varphi, \sin\varphi)^\top$, $\varphi \in [0, 2\pi)$. Compute the gradient $\nabla f(x)$ at x, the scalar product $d \cdot \nabla f(x)$ and the directional derivative $\frac{\partial f}{\partial d}(x)$ by definition as well.

Exercise T47:

(a) Consider the vector valued functions f and g defined as follows:

$$f(x, y) = \begin{pmatrix} xy \\ x^2 + y^2 \end{pmatrix}, \quad g(x, y) = \begin{pmatrix} \cos x \\ \sin y \end{pmatrix}, \quad x, y \in \mathbb{R}.$$

Calculate $(f^\top g)'$ and $(f \circ g)'$.

(b) Determine for the vector valued functions

$$f(x, y) = \begin{pmatrix} xy \\ x^2 - y^2 \end{pmatrix} \quad \text{and} \quad g(x, y) = \begin{pmatrix} y \cosh x \\ y \sinh x \end{pmatrix}, \quad x, y \in \mathbb{R},$$

the derivative $(f \circ g)'$ directly and by the chain rule as well.

Exercise T48:

(a) Determine all partial first derivatives of the function

$$F(x, y) = \int_1^{\sqrt{x}} \frac{1}{\tau} \cos(\tau^2 y \pi) d\tau, \quad x, y \geq 1\,.$$

(b) Compute the value $g'(2)$ for the function $g(t) = \int_1^{\sqrt{t}} \frac{1}{\tau} \cos(\tau^2 t^2 \pi) d\tau, \quad t \geq 1\,.$

Hint: Use $g(t) = F(t, t^2)$ and the chain rule.

Tutorial date: Monday, July 13, 3:45 p.m.–5:15 p.m.