Universität Karlsruhe (TH)
Institut für Algebra und Geometrie
PD Dr. F. Hettlich
Dipl.-Math.techn. A. Schkarbanenko

11	12	13	14	15	Σ

Karlsruhe, 5.5.2009

Student Nr.: ....

Worksheet No. 3
Advanced Mathematics II

Exercise 11: Consider the three matrices

$$A = \begin{pmatrix} 1 & 5 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 0 & 2 \\ 1 & 2 & 3 \end{pmatrix}, \quad C = \begin{pmatrix} 4 & -1 \\ 0 & 3 \\ 2 & 1 \end{pmatrix}.$$

Decide which of the following products are defined and compute them if possible:

$$AB, BA^\top, CA, CA^\top, C^\top A^\top, B^\top C^\top A^\top, (BA^\top)^\top C^\top, (CB)^\top A.$$

Exercise 12: Let

$$x = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} \alpha \\ 2 \end{pmatrix}$$

be vectors in $\mathbb{R}^2$ with a parameter $\alpha \in \mathbb{R}$. We are looking for a matrix $A \in \mathbb{R}^{2\times 2}$, such that $Ax = y$ and $Ay = x$. For which α can such A be uniquely determined? Compute the matrix for this case. Compute also its inverse A^{-1} if it exists.

Exercise 13: For 3 times kitchen duty (K) and once swabbing the deck (S), poor seaman Hein B. obtains one Euro (E) and a pudding (P) from his captain. Moreover, Hein gets a pudding for four hand made fishing rods (R) and once swabbing the deck. For two fishing rods and four fish (F) he can afford a lemonade (L) in the harbour bar. Finally, for α fish and one kitchen duty, his captain tells one of his thrilling cock-and-bull stories (G). Here, $\alpha \in \mathbb{R}$ is a parameter depending on the captain's mood. Find the matrix A_α representing the linear mapping $(E, P, L, G) \mapsto (K, S, R, F)$. For which α is this mapping invertible? If it is invertible, compute the inverse and describe its meaning for kitchen duty, swabbing, fishing rods and fish.

Exercise 14: Let $A = \frac{1}{4}\begin{pmatrix} 1 & -1 & 5 & -1 \\ -1 & 7 & -1 & -1 \\ 5 & -1 & 1 & -1 \\ -1 & -1 & -1 & 7 \end{pmatrix} \in \mathbb{R}^{4\times 4}$, $b_1 = (1,1,1,1)^\top$, $b_2 = (1,0,-1,0)^\top$ and $y = (1,3,5,3)^\top$.

(a) Find b_3 and b_4 such that b_1, b_2, b_3 and b_4 are orthogonal to each other.

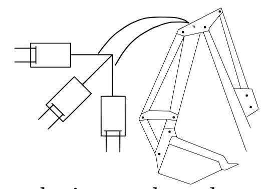
(b) Determine the vectors Ab_k and check if there is an α_k such that $Ab_k = \alpha_k b_k$, $k = 1,2,3,4$.

(c) Write y as a linear combination of the vectors b_k, $k = 1,2,3,4$, i.e. find the presentation $\lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3 + \lambda_4 b_4 = y$. Hint: scalar product and the orthogonality of b_k, $k = 1,2,3,4$.

(d) Write x as a linear combination of vectors b_k, $k = 1,2,3,4$ in $Ax = y$ by using α_k and λ_k .

Exercise 15: *By a measurement rosette (Dehnmessstreifen-Rosette) on a surface the extension state of a hydraulic shovel may be determined in the form of a strain tensor with respect to the x_1, x_2, x_3 coordinate system. We are interested in the principle strains (Hauptdehnungen) as they determine the maximal forces in the corresponding principle strain axes in isotropic materials. For the given tensor ε we will discuss the corresponding linear system with parameter $\lambda \in \mathbb{R}$ given on the right:*

$$\varepsilon = \frac{1}{50}\begin{pmatrix} 142 & -144 & 0 \\ -144 & 58 & 0 \\ 0 & 0 & -125 \end{pmatrix} \quad \rightarrow \quad \begin{array}{rrrcl} 142x_1 & -144x_2 & & = & 50\lambda x_1 \\ -144x_1 & +58x_2 & & = & 50\lambda x_2 \\ & & -125x_3 & = & 50\lambda x_3 \end{array}$$



Determine the principle strains by computing all $\lambda \in \mathbb{R}$ such that the linear system has solutions other than the zero vector. For each λ, give a normed solution vector to the system of length 1 as corresponding principle strain axis. Determine the angles between the principle strain axes, discuss whether they form a basis of $\mathbb{R}^3$ and determine their angle to the standard basis.

Due date: Please hand in your homework on Thursday, May 14, 11:15 a.m.

Tutorial No. 3
Advanced Mathematics II

Exercise T9: Which of the products $AB, AX, BX, X^\top A, (A^\top X)^\top B, B^\top X, XX^\top$ involving the matrices given below are well defined? Where appropriate determine the size of the resulting matrix and then evaluate the product.

$$A = \begin{pmatrix} 3 & 7 \\ 2 & 8 \\ 3 & 4 \end{pmatrix} \qquad B = \begin{pmatrix} -2 & 8 \\ 6 & 5 \end{pmatrix} \qquad X = \begin{pmatrix} 8 \\ 4 \\ 1 \end{pmatrix}.$$

Exercise T10: Calculate the inverses of the square matrices

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

and determine a matrix $C \in \mathbb{R}^{3\times 3}$, such that

$$ACB = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 1 \\ 2 & 3 & 4 \end{pmatrix}.$$

Hint: The matrix multiplication is not *commutative*, i.e. $AB \neq BA$ in general.

Exercise T11: Given the 3 vectors

$$b_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \qquad b_2 = \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}, \qquad b_3 = \begin{pmatrix} 0 \\ -4 \\ 1 \end{pmatrix}$$

in $\mathbb{R}^3$, determine the matrix A of the linear map Φ with $\Phi(e_1) = b_1,\ \Phi(e_2) = b_2,\ \Phi(e_3) = b_3$, where e_j denotes the jth coordinate unit vector. Also given the 3 vectors

$$c_1 = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}, \qquad c_2 = \begin{pmatrix} 3 \\ 0 \\ -4 \end{pmatrix}, \qquad c_3 = \begin{pmatrix} -5 \\ 0 \\ 7 \end{pmatrix},$$

show that the matrix

$$B = \begin{pmatrix} 0 & 1 & 0 \\ 7 & 4 & 5 \\ 4 & 2 & 3 \end{pmatrix}$$

defines a linear map $\Psi : x \mapsto Bx$, with $Bc_j = e_j$, $j = 1, 2, 3$. Also show that the linear map $\Lambda : x \mapsto (AB)x$ satisfies $\Lambda c_j = b_j$, $j = 1, 2, 3$.

Exercise T12: Consider the vectors $x^{(1)} = (1, 0, 0)^\top, x^{(2)} = (1, 0, 1)^\top, x^{(3)} = (1, -1, 0)^\top$ and $y^{(1)} = (1, 2, 3)^\top, y^{(2)} = (2, -2, 7)^\top, y^{(3)} = (-1, 0, -3)^\top$. Determine a matrix A such that $Ax^{(i)} = y^{(i)}$ for $i \in \{1, 2, 3\}$.

Tutorial date: Monday, May 11, 3:45 p.m.-5:15 p.m.