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Karlsruhe, 19.5.2009

Student Nr.: ....

Worksheet No. 5
Advanced Mathematics II

Exercise 21: Let

$$C = \begin{pmatrix} 5 & 4 & -5 & 5 \\ -1 & 1 & 2 & -2 \\ -3 & -4 & 7 & -5 \\ -4 & -5 & 7 & -5 \end{pmatrix}.$$

Calculate all eigenvalues and eigenvectors of C.

Exercise 22: Given two planes $E:\ 2x_1 - x_2 - 2x_3 = 0$ and $F:\ x(\lambda,\mu) = (0,1,0)^\top + \lambda(4,1,3)^\top + \mu(4,-7,3)^\top$, $\lambda, \mu \in \mathbb{R}$. Let the linear mapping $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ be a reflection in E and the mapping $\Psi : \mathbb{R}^3 \to \mathbb{R}^3$ the orthogonal projection on F.

(a) Determine the matrices A of Φ and B of Ψ with respect to the standard basis. Check if A or B are orthogonal matrices and compute the matrix products A^2 as well as B^2.

(b) Compute the eigenvalues and eigenvectors of Φ and Ψ.

Exercise 23: The linear mappings $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ and $\Psi : \mathbb{R}^3 \to \mathbb{R}^3$ are represented by the following matrices with respect to the standard basis:

$$A = \frac{1}{3}\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}, \quad B = \frac{1}{9}\begin{pmatrix} 7 & 4 & -4 \\ 4 & 1 & 8 \\ -4 & 8 & 1 \end{pmatrix}.$$

(a) Compute the eigenvalues and eigenvectors of A and B.

(b) Check if Φ and Ψ describe a projection, reflection or rotation. If so, find the projection plane, reflection plane or rotation axis and angle of rotation.

Exercise 24:

(a) Determine the eigenvalues of $A = \begin{pmatrix} 5 & 0 & 4 \\ 0 & -6 & 0 \\ 1 & 0 & 2 \end{pmatrix}$.

(b) Let $p(x) = c_3x^3 + c_2x^2 + c_1x + c_0$ be the characteristic polynomial of A. Show that $p(A) = 0$, i.e. $c_3A^3 + c_2A^2 + c_1A + c_0I_3 = 0$. (This is true in general for every square matrix and its characteristic polynomial!)

(c) From $p(A) = 0$, determine the inverse matrix A^{-1}.

Exercise 25: *In physical chemistry the molecular orbitals and the corresponding energies of π-electron systems can be computed quantum mechanically via the Hückel method. The molecular structure is represented in a kind of normed adjacency matrix, where every carbon compound is represented by a 1. The eigenvalues of this matrix characterise the energies, and the eigenvectors describe the structure of the molecular orbitals.*

Compute the eigenvalues and eigenvectors of the matrix M for the illustrated cyclopropenyl compound:

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  C
 / \        →
C — C
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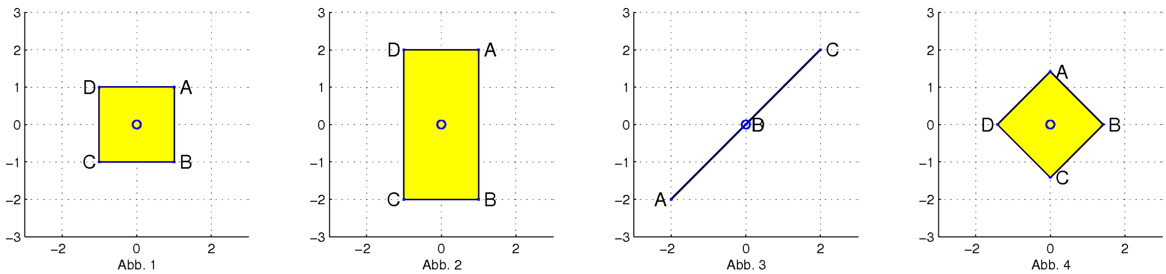
$$M = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Due date: Please hand in your homework on Thursday, May 28, 11:15 a.m.

Tutorial No. 5
Advanced Mathematics II

Exercise T17: Figure 1 shows a square around the origin with the vertices A, B, C, D. The following figures show the images of the square obtained by applying different linear mappings $\Phi_{1,2,3} : \mathbb{R}^2 \to \mathbb{R}^2$:



Determine the eigenvalues of the linear mappings. Sketch the eigenvectors if possible.

Exercise T18:

(a) Compute the characteristic polynomial $p(\lambda) = \det(A - \lambda I)$ of A for $\lambda \in \mathbb{R}$, where

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad A = \begin{pmatrix} 2 & 1 & 2 \\ 1 & 2 & 2 \\ 1 & 1 & 3 \end{pmatrix}.$$

(b) Determine the eigenvalues, i.e. zeros $\lambda_1, \lambda_2, \lambda_3$ of the polynomial $p(\lambda)$.

(c) Find the eigenvectors, i.e. the solution set of the homogeneous system of linear equation $(A - \lambda_j I)x = 0$ for $j = 1, 2, 3$.

Exercise T19: Let the linear mapping $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ be a rotation around the x_2-axis by the angle $\frac{\pi}{3}$ and let the linear mapping $\Psi : \mathbb{R}^3 \to \mathbb{R}^3$ be a reflection on the plane $E:\ x_1 + 2x_2 = 0$.

(a) Determine the matrices A of Φ and B of Ψ with respect to the standard basis. Check if A and B are orthogonal matrices.

(b) Compute the eigenvalues of A and B, as well as the eigenvectors to the real eigenvalues.

Exercise T20: Determine all eigenvalues of the square matrices

$$\text{(a)} \quad A = \begin{pmatrix} 2 & 7 & -2 \\ -1 & -1 & -1 \\ -1 & -5 & 3 \end{pmatrix}, \qquad \text{(b)} \quad B = \frac{1}{2}\begin{pmatrix} 1 & -5 & 5 \\ 5 & -9 & 5 \\ 5 & -5 & 1 \end{pmatrix},$$

and an eigenvector to every real eigenvalue λ.

Tutorial date: Monday, May 25, 3:45 p.m.–5:15 p.m.