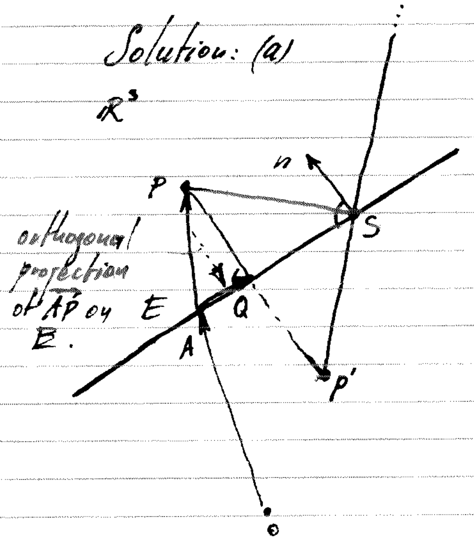
4th exercise: A laser beam L is generated at point $P = (11|15|10)$ and is directed by vector $D = (-5,-9,-5)^T$. This beam hits a mirror plane $E := \{x \in \mathbb{R}^3 : x_1 + 8x_2 + 4x_3 = 9\}$ at the point S.

(a) Determine the distance between P and E.

(b) Find the point S where L hits the plane E.

(c) Give a parameter form of the reflected beam L'.

Solution: (a)



$d(P,E) = ?$ (distance between P and E).

Idea: Find $Q \in E \mathrel{\hat{=}}$ the orthogonal projection of P on E the plane.

1) We need the normal vector n:

$n = (1,8,4)^T$. (see the normal form of E)

$(\vec{OQ} = \underbrace{\vec{OA}}_{=(a_1|a_2|a_3)^T} + \vec{AQ})$ (compare page 17, projection point)

2) A: choose two of three components arbitrary for example $a_2 = 0$, $a_3 = 0$ and compute a_1 by inserting a_2, a_3 into the normal form of E.
We get: $A = (9|0|0)$.

(Important property of $\vec{QP}$: $\vec{QP} = t\vec{n}$ for a $t \in \mathbb{R}$.) $(*)$
(" " " $\vec{AQ}$: $\vec{AQ} \perp n$.)

3) Determine $\vec{QP}$: $\vec{AQ} \perp n \Leftrightarrow \vec{AQ} \cdot n = 0 \Leftrightarrow (\vec{AP} + \vec{PQ}) \cdot n = 0 \Leftrightarrow$

$\Leftrightarrow \vec{AP} \cdot n = -\vec{PQ} \cdot n \Leftrightarrow \vec{AP} \cdot n = -\vec{QP} \cdot n$

$\overset{(*)}{\Leftrightarrow} \vec{AP} \cdot n = -t \underbrace{n \cdot n}_{= \|n\|^2 \neq 0} \Leftrightarrow \boxed{\frac{\vec{AP} \cdot n}{\|n\|^2} = t} \overset{(*)}{\Rightarrow} \vec{QP} = \underbrace{\frac{\vec{AP} \cdot n}{\|n\|^2} \cdot n}_{\text{orthogonal projection of } \vec{AP} \text{ onto } E. \text{ (Page 17)}}$

$$\vec{QP} = \frac{\vec{AP} \cdot n}{\|n\|^2} \cdot n = \frac{\left[-\begin{pmatrix}9\\0\\0\end{pmatrix} + \begin{pmatrix}11\\15\\10\end{pmatrix}\right] \cdot \begin{pmatrix}1\\8\\4\end{pmatrix}}{1^2 + 64 + 16} * \begin{pmatrix}1\\8\\4\end{pmatrix} = \frac{\begin{pmatrix}2\\15\\10\end{pmatrix} \cdot \begin{pmatrix}1\\8\\4\end{pmatrix}}{81} \cdot \begin{pmatrix}1\\8\\4\end{pmatrix} = \underbrace{\frac{2+120+40}{81}}_{=2} \begin{pmatrix}1\\8\\4\end{pmatrix} = \begin{pmatrix}2\\16\\8\end{pmatrix}.$$

$$\Rightarrow d(P,E) = \|\vec{QP}\| = 2 \cdot \|\vec{n}\| = 2 \cdot \sqrt{1 + 64 + 16} = 2 \cdot \sqrt{81} = \underline{18}.$$

end of (a)

(b) $L \cap E =: S$. (cross section point)

The parameter form of the line L:

$$L: x(\lambda) = \begin{pmatrix} 11 \\ 15 \\ 10 \end{pmatrix} + \lambda \begin{pmatrix} -5 \\ -7 \\ -5 \end{pmatrix} = \begin{pmatrix} 11-5\lambda \\ 15-7\lambda \\ 10-5\lambda \end{pmatrix} \begin{matrix} \hat{=}\, x_1 \\ \hat{=}\, x_2 \\ \hat{=}\, x_3 \end{matrix} \quad \lambda \geq 0.$$

Insert the coordinates x_1, x_2, x_3 into the normal form of E and find λ:

$$(11-5\lambda)\cdot 1 + (15-7\lambda)\cdot 8 + (10-5\lambda)\cdot 4 = 9.$$

$$\Leftrightarrow \quad [\underbrace{11+120+40-9}] = (5+56+20)\lambda$$

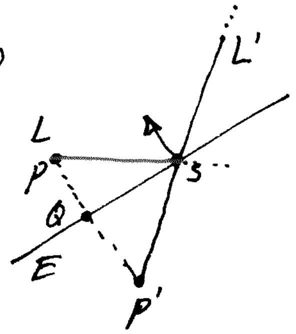
$$162 = 81\lambda$$

$$2 = \lambda. \quad , \lambda \geq 0 \checkmark.$$

Plug $\lambda = 2$ into the parameter form of L:

$$x(2) = \begin{pmatrix} 11 \\ 15 \\ 10 \end{pmatrix} + 2\begin{pmatrix} -5 \\ -7 \\ -5 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \vec{OS}.$$

(c)



Idea: "Mirror" P on E, we get P'. Then $\vec{P'S}$ is the direction of L'. The beam L' is starting at S. That's all!

(1) P': $\vec{OP'} = \vec{OP} + 2\vec{PQ} \overset{(a)}{=} (11,15,10)^T + 2(-2,-16,-8)^T$

$= (7,-17,-6)^T$.

(2) $\vec{P'S} = \vec{OS} - \vec{OP'} \overset{(b)}{=} (1,1,0)^T - (7,-17,-6)^T =$

$= (-6,+16,6)^T$.

So: $L': x(\lambda) = \vec{OS} + \lambda \cdot \vec{P'S} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -6 \\ 16 \\ 6 \end{pmatrix}, \quad \lambda \geq 0.$

——— end of 4th exercise ———

2. Problem session

$M=\{u,v,w\}$ linearly independent:
$\alpha_1 u + \alpha_2 v + \alpha_3 w = 0$
has only the "trivial" solution
$(\alpha_1,\alpha_2,\alpha_3) = (0,0,0)$.

and

M span a subspace U means:
$\alpha_1 u + \alpha_2 v + \alpha_3 w = x$
has a solution for all $x \in U$

$\Updownarrow$

M is a basis of U:
$\alpha_1 u + \alpha_2 v + \alpha_3 w = x$
has a unique solution
for all $x \in U$.