4th Exercise: (Compare 15th Exercise on 3rd Problem Sheet).

Let
$$\mathcal{E} = \frac{1}{50}\begin{pmatrix} 142 & -144 & 0 \\ -144 & 58 & 0 \\ 0 & 0 & -125 \end{pmatrix} \in \mathbb{R}^{3\times 3}.$$

(a) Compute $\det(\mathcal{E}-\lambda I_3)$ for $\lambda \in \mathbb{R}$.

(b) For which values of $\lambda \in \mathbb{R}$ the system of linear equations $\mathcal{E}x = \lambda x$ has nontrivial solution $x \in \mathbb{R}^3$?

Solution:

(a) $$\det\big(\underbrace{\mathcal{E}-\lambda I_3}_{\in\mathbb{R}^{3\times 3}}\big) = \det\left(\frac{1}{50}\begin{pmatrix} 142-50\lambda & -144 & 0 \\ -144 & 58-50\lambda & 0 \\ 0 & 0 & -125-50\lambda \end{pmatrix}\right)$$

$$= \left(\frac{1}{50}\right)^3 \cdot (-125-50\lambda)\cdot(-1)^{3+3}\cdot \det\begin{pmatrix} 142-50\lambda & -144 \\ -144 & 58-50\lambda \end{pmatrix} =$$

expansion rule and argument above!

$$= \frac{1}{50^3}(-125-50\lambda)\big((142-50\lambda)(58-50\lambda)-(-144)(-144)\big)$$

$$= \frac{1}{50^3}(\underbrace{-125-50\lambda}_{50})\big[\underbrace{50^2\lambda^2 - 200\cdot 50\lambda - (\underbrace{142\cdot 58-144^2}_{=12500})}_{50^2}\big]$$

$$= \left(-\frac{5}{2}-\lambda\right)(\lambda^2-4\lambda-5) = \left(-\frac{5}{2}-\lambda\right)(\lambda-5)(\lambda+1), \ \lambda\in\mathbb{R}. \qquad (*)$$

(b) $\mathcal{E}x = \lambda x \iff \mathcal{E}x - \lambda x = 0 \iff \mathcal{E}x - \lambda I x = 0 \iff (\mathcal{E}-\lambda I)x = 0$

This system of lin equations has nontrivial solution if and only if the matrix $\mathcal{E}-\lambda I$ is singular (Thm 1.33). By Thm 1.43(f) this is equivalent to $\det A = 0$.

~~So~~ Thus, for $\lambda \in \left\{-\frac{5}{2}, 5, -1\right\}$ — That are the zeros of the polynomial $p(\lambda) := \left(-\frac{5}{2}-\lambda\right)(\lambda-5)(\lambda+1)$, see $(*)$.

this system has nontriv. solution $x \in \mathbb{R}^3$.

Such solutions we call Eigenvectors and the corresponding λ "Eigenvalues".
That's the next topic in AML!

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