Solving linear homogeneous system of diff. equations with constant coefficients.

hom. system

$u'(x) = Au(t)$, $A \in \mathbb{C}^{n \times n}$. Note: u is n-dim, so u' too! Thus A is a square matrix.

Ausatz: $u(x) = e^{\lambda x} \cdot v$, $\lambda \in \mathbb{C}$, $v \in \mathbb{C}^n$, λ, v to find.

$\lambda v = Av \iff (A - \lambda I)v = 0$

$p(\lambda) = \det(A - \lambda I)$ (characteristic polynomial)
$\det(B) \neq 0$ if and only if B is a regular matrix.

$\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{C}$ eigenvalues, not always pairwise different

$(A - \lambda_k E) v_k = 0$ Determine the eigenvectors
(λ_k, λ_j different, so v_k, v_j lin. independent.)

Awatz

Solution: $u_n(x) = c_1 \cdot e^{\lambda_1 x} v_1 + \dots + c_n \cdot e^{\lambda_n x} v_n$.

(The solution is a n-dim vector space).

3rd Exercise: (a) Find the general solution of the linear system of differential equation

$$u'(t) = \overbrace{\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{pmatrix}}^{=:A} u(t), \quad t \in \mathbb{R}_{\geq 0}$$

(b) Determine the real Solution of the initial value problem

$$u'(t) = A \cdot u(t), \quad u(0) = (2,-2,3)^T, \quad t \in \mathbb{R}_{\geq 0}.$$

Solution: (a) Show the slide.

- Eigenvalues: $p(\lambda) = \det(A - \lambda I_3) \underset{\text{rule}}{\overset{\text{expansion}}{=}} (1-\lambda)(\lambda^2 - 2\lambda + 5) \underset{\text{the square}}{\overset{\text{completing}}{=}}$
$= (1-\lambda)(\lambda - 1 - 2i)(\lambda - 1 + 2i)$.

Thus: $\lambda_1 = 1$, $\lambda_2 = 1+2i$, $\lambda_3 = 1-2i$.

- Eigenvectors: Following systems to solve: $(A - \lambda_k I_3) y_k = 0.$, $k = 1,2,3$.
By Gaussian elimination we obtain: (for example):

$y_1 = (2,-3,2)^T$, $y_2 = (0, i, 1)^T$, $y_3 = \overline{y_2}$, because:

$\lambda_3 = \overline{\lambda_2}$. $A y_2 = \lambda_2 y_2 \Rightarrow \overline{A y_2} = \overline{\lambda_2 y_2} = \lambda_3 \overline{y_2}$ (*)

A is real, so $\overline{A} = A$. That means: $\overline{A y_2} = A \overline{y_2} \overset{(*)}{=} \lambda_3 \overline{y_2} \Rightarrow$

$\Rightarrow \overline{y_2}$ is eigenvector for λ_3. Set $y_3 := \overline{y_2}$. We have done!

- Complex fundamental system is

$$\left\{ y_1 e^{\lambda_1 t}, y_2 e^{\lambda_2 t}, y_3 e^{\lambda_3 t} \right\} = \left\{ \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix} e^t, \begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix} e^{(1+2i)t}, \begin{pmatrix} 0 \\ -i \\ 1 \end{pmatrix} e^{(1-2i)t} \right\}.$$

- Real fundamental system (see the calculation in 2nd exercise):

$$\left\{ \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix} e^t, e^t \begin{pmatrix} 0 \\ -\sin 2t \\ \cos 2t \end{pmatrix}, e^t \begin{pmatrix} 0 \\ \cos 2t \\ \sin 2t \end{pmatrix} \right\}.$$

- The general real solution of the system is:

$$u(t) = c_1 \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix} e^t + c_2 \begin{pmatrix} 0 \\ -\sin 2t \\ \cos 2t \end{pmatrix} e^t + c_3 \begin{pmatrix} 0 \\ \cos(2t) \\ \sin(2t) \end{pmatrix} e^t, \quad c_1, c_2, c_3 \in \mathbb{R}!$$

(b) Initial value problem: (Thm. 2.9 → This IVP has exactly one solution (it is even cont. diff.)

$$u(0) = c_1 \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}.$$

This system of lin. equations has triangle structure already!

$$\left(\begin{array}{ccc|c} 2 & 0 & 0 & 2 \\ -3 & 0 & 1 & -2 \\ 2 & 1 & 0 & 3 \end{array}\right) \begin{array}{ll} \Rightarrow c_1 = 1 & \\ \Rightarrow -3 + c_3 = -2 & \Rightarrow c_3 = 1 \\ \Rightarrow 2 + c_2 = 3 & \Rightarrow c_2 = 1. \end{array}$$

The unique solution of the IVP is then

$$u(t) = \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ -\sin(2t) \\ \cos(2t) \end{pmatrix} e^t + \begin{pmatrix} 0 \\ \cos(2t) \\ \sin(2t) \end{pmatrix} e^t, \quad t \geq 0.$$

End.