Problem Class No. 10

1.[st] Exercise: Calculate the Laplace-transform of $f(t) = t \cdot e^{\lambda t}$, $t \geq 0$, $\lambda \in \mathbb{R}$.

2.[nd] Exercise: Determine the Laplace-transform of

(a) $f(t) = e^{3t} \cdot t$ applying damping theorem

(b) $f(at) = e^{3at} \cdot at$

(c) $f(t) = e^{3t} \cdot t$ applying differentiation in image space.

3[rd] Exercise: Determine the Laplace-transform of

(a) $f(t) = t, \quad t \geq 0$

(b) $g(t) = \begin{cases} 0, & t < a = 1 \\ (t-a), & t \geq a = 1 \end{cases}$ and $h(t) = \begin{cases} 0, & t < a = 1 \\ 1, & t \geq a = 1 \end{cases}$

(c) $u(t) = \begin{cases} 0, & t < a = 1 \\ t, & t \geq a = 1 \end{cases}$

4[th] Exercise: Solve the initial value problem

$$y''(t) + y(t) = 0, \quad y(0) = A, \; y'(0) = B, \quad t \geq 0$$

using the Laplace-transform.

5[th] Exercise: Determine the Solution of the initial value problem

$$u'(t) - 6\,u(t) + 3v(t) = 8e^{t}$$

$$v'(t) - 2u(t) - v(t) = 4e^{t}$$

$$u(0) = -1, \quad v(0) = 0, \quad t \geq 0.$$

5^{th} Exercise: System of differential equation.

Solution: One possibility to solve this problem is as in Exercise 35, 34, 33

~~Then~~ Another one is the application of Laplace-transform!

Set $U(s) = (\mathcal{L}u)(s)$ and $V(s) = (\mathcal{L}v)(s)$.

By theorem 3.12 (Differentiation in original space) we have

- $\mathcal{L}(u'(t))(s) = sU(s) - u(0) = sU(s) + 1$.
- $\mathcal{L}(v'(t))(s) = sV(s) - v(0) = sV(s)$.

Put this into the given system (transform the right hand side):

$$\begin{cases} sU(s) + 1 - 6U(s) + 3V(s) = 8\frac{1}{s-1} & (1) \\ sV(s) - 2U(s) - V(s) = 4\frac{1}{s-1} & (2) \end{cases}$$

$$\begin{cases} U(s)(s-6) + 3V(s) = 8\frac{1}{s-1} - 1 & \cdot (s-1) \\ U(s)(-2) + (s-1)V(s) = 4\frac{1}{s-1} & \cdot (s-1) \end{cases}$$

$$\begin{cases} U(s)(s-6)(s-1) + 3(s-1)V(s) = 8 - s + 1 \\ U(s)(-2)(s-1) + (s-1)^2 V(s) = 4. \end{cases}$$

Table:

$$\left(\begin{array}{cc|c} (s-6)(s-1) & 3(s-1) & 9-s \\ -2(s-1) & (s-1)^2 & 4 \end{array}\right) \begin{array}{l} \cdot 2 \\ \cdot (s-6) \\ s \neq 6 \end{array} \rightsquigarrow \left(\begin{array}{cc|c} 0 & 6(s-1) + (s-1)^2(s-6) & 2(9-s) + 4(s-6) \\ -2(s-1) & (s-1)^2 & 4 \end{array}\right)$$

$$\to \left(\begin{array}{cc|c} 0 & (s-1)\overbrace{(6 + (s-1)(s-6))}^{=(s-3)(s-4)} & 2(s-3) \\ -2(s-1) & (s-1)^2 & 4 \end{array}\right) \cdot \frac{-(s-1)}{(s-3)(s-4)}, \quad s \neq 3, s \neq 4.$$

$$\to \left(\begin{array}{cc|c} 0 & (s-1)(s-3)(s-4) & 2(s-3) \\ -2(s-1) & 0 & 4 - \frac{2(s-1)}{s-4} \end{array}\right)$$

$$\Rightarrow V(s) = \frac{2\cancel{(s-3)}}{(s-1)\cancel{(s-3)}(s-4)} = \frac{2}{(s-1)(s-4)} \overset{\text{Partial fraction decomp.}}{=} -\frac{2}{3}\frac{1}{s-1} + \frac{2}{3}\frac{1}{s-4}, \quad s > 4.$$

$$U(s) = \frac{\cancel{4}^{\,2}}{\cancel{-2}_{\,1}(s-1)} - \frac{\cancel{2(s-1)}}{\cancel{-2(s-1)}(s-4)} = \underbrace{-\frac{2}{s-1}}_{= -2\cdot\mathcal{L}(e^t)(s)} + \underbrace{\frac{1}{s-4}}_{= \mathcal{L}(e^{4t})(s).}, \quad s > 4.$$

Thus $u(t) = -2e^t + e^{4t}$, $t \geq 0$, $v(t) = -\frac{2}{3}e^t + \frac{2}{3}e^{4t}$. $t \geq 0$.