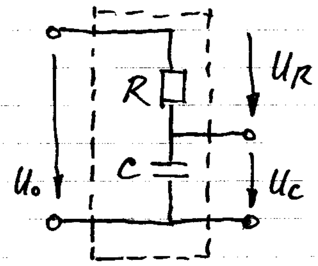
4th Exercise:



Consider a series connection of an electrical condenser with capacity C and a resistor R. Let the voltage $U_0 = U_0(t)$. Our goal is to determine the output signal $U_c = U_c(t)$.

(a) There holds: $U_0 = U_R + U_c$, $U_R = R \cdot I$, $I = Q'$, $Q = C \cdot U_c$

Set up the differential equation for U_c

(b) Let $U_0(t) = \delta(t - t_0)$. Determine by means of Laplace-transform the function $K : [0, \infty) \to \mathbb{R}$ with: $U_c(t) = K(t - t_0)$ for $t \geq t_0 > 0$

K is called "pulse response".

(c) Show that $U_c(t) = U_0(t) * K(t)$.

(d) Compute $U_c(t)$ for $R = 10\,k\Omega$, $C = 100\,\mu F$ and $U_0(t) = \cos(\omega t)$.

4th Exercise: $\overset{\text{given}}{U_0(t)} = U_R(t) + U_C(t) = R \cdot I(t) + U_C =$

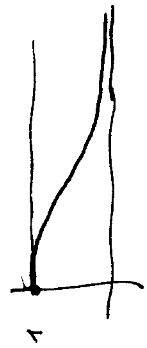
(a) $= R \cdot Q'(t) + U_C(t) = R \cdot C \cdot U_C'(t) + U_C(t)$.

(b) Apply the Laplace-transform on DE in (a): with initial cond: $U_C(0) = U_R(0) = 0$.

$$R \cdot C \cdot s \cdot (\mathcal{L} U_C)(s) + (\mathcal{L} U_C)(s) = (\mathcal{L} U_0)(s).$$

$$\Leftrightarrow (\mathcal{L} U_C)(s) = (\mathcal{L} U_0)(s) \cdot \frac{1}{RCs+1} = (\mathcal{L} U_0)(s) \cdot \underbrace{\frac{1}{RC} \cdot \frac{1}{s + \frac{1}{RC}}}_{\text{transfer function.}} =$$

$$= (\mathcal{L} U_0)(s) \cdot \mathcal{L}\left(\frac{1}{RC} \cdot e^{-t/RC}\right)(s). \quad (*)$$



- The Laplace-transform of $U_0(t) = \delta(t - t_0)$ is $(\mathcal{L} U_0)(s) = e^{-t_0 s}$, $s > 0$ (Def 3.23.)

put it into (*) $\Rightarrow (\mathcal{L} U_C)(s) = e^{-t_0 s} \cdot \mathcal{L}\left(\frac{1}{RC} \cdot e^{-t/RC}\right)(s)$.

Shifting Theorem $\Rightarrow U_C(t) = \begin{cases} 0, & 0 \le t < t_0 \\ \frac{1}{RC} \cdot e^{-\frac{(t-t_0)}{RC}}, & t \ge t_0 \end{cases}$ $\Rightarrow K(t) = \frac{1}{RC} \cdot e^{-\frac{t}{RC}}$, "pulse response."

(c) From (*) we have $(\mathcal{L} U_C)(s) = (\mathcal{L} U_0)(s) \cdot (\mathcal{L} K)(s) \Rightarrow$

$\underset{\text{Convolution Thm}}{\Rightarrow} U_C(t) = U_0(t) * K(t)$.

Note: $\int_0^\infty K(t)\,dt = [-e^{-t/RC}]_0^\infty = 1 \Rightarrow U_C$ is a ...

(d) $R \cdot C = 10k\Omega \cdot 100\mu F = 1000 k\Omega \cdot \mu F = 1 \Omega F = 1 s^{-1}$?

$U_0(t) = \cos(\omega t)$.

$$\Rightarrow U_C(t) = U_0(t) * K(t) = \int_0^t U_0(t-\tau) \cdot K(\tau)\,d\tau =$$

$$= \int_0^t \underbrace{\cos(\omega(t-\tau))}_{u} \cdot \underbrace{e^{-\tau}}_{v'}\,d\tau \overset{\text{Int. by parts}}{=}$$

$$= [-\cos(\omega(t-\tau)) \cdot e^{-\tau}]_0^t - \int_0^t \underbrace{-\sin(\omega(t-\tau))}_{u} \underbrace{e^{-\tau}}_{v'}\,d\tau \overset{\text{Int by parts}}{=}$$

$$= -e^{-t} + \cos(\omega t) + [-\omega \sin(\omega(t-\tau)) \cdot e^{-\tau}]_0^t - \omega \int_0^t \cos(\omega(t-\tau)) \cdot (-\omega) \cdot (-e^{-\tau})\,d\tau$$

$$= -e^{-t} + \cos(\omega t) + \omega \sin(\omega t) - \omega^2 U_C(t).$$

$$\Rightarrow U_C(t) = \frac{1}{1+\omega^2} \cdot \Big(\underbrace{-e^{-t}}_{\to 0 \text{ for } t \to \infty} + \underbrace{\cos(\omega t) + \omega \sin(\omega t)}_{\text{oscillation with amplitude of } \sqrt{1+\omega^2}}\Big)$$

$\to$ complete amplitude: $\frac{1}{1+\omega^2} \cdot \sqrt{1+\omega^2} = \frac{1}{\sqrt{1+\omega^2}}$.

The higher ~~the amplitude~~ frequency, the lower ~~the~~ amplitude $\Rightarrow$ lowpass filter!