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Karlsruhe, 4 May 2010

16	17	18	19	20	$\sum$

Student Nr.: ....

Worksheet No.4
Advanced Mathematics II

Exercise 16: Which of the products $AB, AX, BX, X^\top A, (A^\top X)^\top B, B^\top X, XX^\top$ involving the matrices given below are well defined? Where appropriate determine the size of the resulting matrix and then evaluate the product.

$$A = \begin{pmatrix} 3 & 7 \\ 2 & 8 \\ 3 & 4 \end{pmatrix} \qquad B = \begin{pmatrix} -2 & 8 \\ 6 & 5 \end{pmatrix} \qquad X = \begin{pmatrix} 8 \\ 4 \\ 1 \end{pmatrix}.$$

Exercise 17: Given the matrices

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 1 \\ 2 & 3 & 4 \end{pmatrix},$$

calculate the inverses of A and B and find a matrix $D \in \mathbb{R}^{3\times 3}$, such that $ADB = C$.

Hint: The matrix multiplication is not *commutative*, i.e. $AB \neq BA$ in general.

Exercise 18: Given the 3 vectors $b_1 = (1,0,1)^\top, b_2 = (-3,2,1)^\top, b_3 = (0,-4,1)^\top$ in $\mathbb{R}^3$, determine the matrix A of the linear map Φ with $\Phi(e_1) = b_1,\ \Phi(e_2) = b_2,\ \Phi(e_3) = b_3$, where e_j denotes the jth coordinate unit vector.

Also given the 3 vectors $c_1 = (-2,1,2)^\top, c_2 = (3,0,-4)^\top, c_3 = (-5,0,7)^\top$. show that the matrix

$$B = \begin{pmatrix} 0 & 1 & 0 \\ 7 & 4 & 5 \\ 4 & 2 & 3 \end{pmatrix}$$

defines a linear map $\Psi : x \mapsto Bx$, with $Bc_j = e_j$, $j = 1,2,3$. Also show that the linear map $\Lambda : x \mapsto (AB)x$ satisfies $\Lambda c_j = b_j$, $j = 1,2,3$.

Exercise 19: Given be two matrices representing a reflection and a rotation, respectively, namely

$$S = \frac{1}{3}\begin{pmatrix} 2 & -2 & 1 \\ -2 & -1 & 2 \\ 1 & 2 & 2 \end{pmatrix} \quad \text{and} \quad R = \frac{1}{2}\begin{pmatrix} 1 & 0 & \sqrt{3} \\ 0 & 2 & 0 \\ -\sqrt{3} & 0 & 1 \end{pmatrix}.$$

(a) Show that S and R are orthogonal matrices.

(b) Determine the plane of reflection associated with S.

(c) Determine the axis of rotation and the angle of rotation associated with R. Hint: $\arccos(1/2) = \frac{\pi}{3}$.

Exercise 20: Given the matrices

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & -2 & 0 \\ 3 & 0 & 1 \end{pmatrix}, \quad R = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 2 & -4 \\ 3 & 6 & 4 \end{pmatrix},$$

(a) show $A = L \cdot R$.

(b) Now set $b = (3,6,-11)^\top$. Solve the linear system $Ax = b$, by calculating a vector y which satisfies $Ly = b$, and then a vector x with $Rx = y$.

Due date: Please hand in your homework until Friday, 14 May, 12:00 into the AM2-box near seminar room Z1, building 01.85 (Fritz-Erler-Str. 1–3).

Tutorial No.4
Advanced Mathematics II

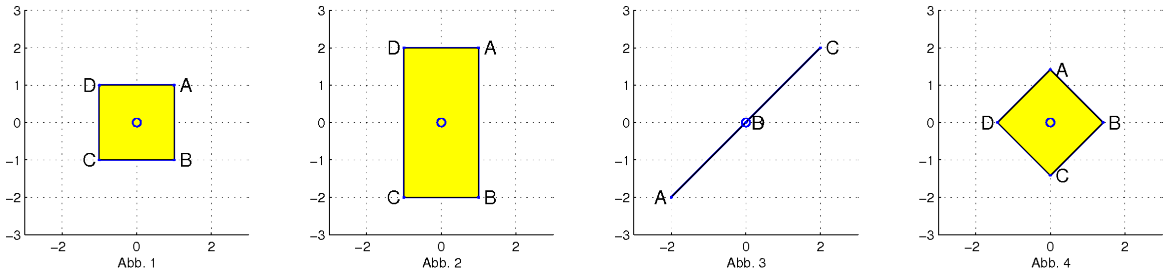
Exercise T10: Consider the three matrices

$$A = \begin{pmatrix} 1 & 5 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 0 & 2 \\ 1 & 2 & 3 \end{pmatrix}, \quad C = \begin{pmatrix} 4 & -1 \\ 0 & 3 \\ 2 & 1 \end{pmatrix}.$$

Decide which of the following products are defined and compute them if possible:

$$AB, BA^\top, CA, CA^\top, C^\top A^\top, B^\top C^\top A^\top, (BA^\top)^\top C^\top, (CB)^\top A.$$

Exercise T11: Figure 1 shows a square around the origin with the vertices A, B, C, D. The following figures show the images of the square obtained by applying different linear mappings $\Phi_{1,2,3} : \mathbb{R}^2 \to \mathbb{R}^2$:



Determine the matrices corresponding to the linear mappings. Describe in each case the inverse map (if it exists). Caluculate the inverse matrices (if possible).

Exercise T12:

(a) Find all fixed points of the matrix

$$A = \begin{pmatrix} -2 & 2 & 1 \\ 2 & -4 & 3 \\ 1 & 3 & -3 \end{pmatrix},$$

(i.e. all $x \in \mathbb{R}^3$ which satisfy $Ax = x$) using Gaussian elimination.

(b) The fixed points of A all lie on a straight line g. Find a vector w orthogonal to g. Calculate the angle between Aw and g.

(c) Let w be an arbitrary vector orthogonal to g. Show that then also Aw is orthogonal to g.

(d) Give a reason why $x \mapsto Ax$ cannot be a rotation around g.

Tutorial date: Friday 7 May