Karlsruhe Institute of Technology
Institute for Applied and Numerical Mathematics
Institute for Algebra and Geometry
PD Dr. G. Thäter
Dipl.-Math. S. Heumann
Dr. K. Kremer
Dipl.-Math.techn. S. Schmitt

Karlsruhe, 15 June 2010

41	42	43	44	45	$\sum$

Student Nr.: ....

Worksheet No.9
Advanced Mathematics II

Exercise 41: Solve the initial value problem

$$(x^2+2x+2)y''(x)+2(x+1)y'(x)-2y(x)=0, \quad y(-1)=1, \quad y'(-1)=0$$

with the power series method. Determine the coefficients explicitly.

Exercise 42: The general solution of the differential equation

$$(2+x)y''(x)+y'(x)=1$$

can be represented by a power series. Determine its radius of convergence.

Exercise 43: Determine the general solution of the differential equation

$$x^2y''(x)+x^3y'(x)-6y(x)=0\,.$$

Use the generalized power series representation $y(x)=\sum_{k=0}^{\infty} a_k x^{k+\lambda}$.
If the solution contains an infinite power series it is here sufficient to define its coefficients recursively.

Exercise 44: We solve Legendre's differential equation

$$(1-x^2)f''(x)-2xf'(x)+n(n+1)f(x)=0, \quad n\in\mathbb{N}.$$

from Exercise 33 with the power series representation $f(x)=\sum_{k=0}^{\infty} a_i x^i$.

(a) Determine the solution for $n=2$ and the initial values $f(0)=1$ and $f'(0)=0$.

(b) Show: For even $n\geq 2$ and the initial values $f(0)=1$ and $f'(0)=0$ we have $a_i=0$ for all $i>n$. We thus get as solution a polynomial of degree n (a so-called *Legendre polynomial*).

(c) Which initial values can we choose for odd n such that the power series stops and yields a polynomial solution?

Exercise 45: We solve the initial value problem $y'(x)=y(x)$, $y(0)=1$ approximately with step size $h>0$. As usual $x_k=x_0+kh$ and y_k denotes the approximate solution obtained from the method at x_k.

(a) Calculate y_k explicitly using the Euler's method.

(b) An *improved Euler's method* for the IVP $y'(x)=f(x,y(x))$, $y(x_0)=y^{(0)}$ can be defined by

$$y_0=y^{(0)}, \qquad y_{k+1}=y_k+\frac{h}{2}\Big(f(x_k,y_k)+f\big(x_{k+1},y_k+hf(x_k,y_k)\big)\Big).$$

Show that using this method one obtains $y_k=(1+h+h^2/2)^k$.

(c) Compare both approximate solutions with the Taylor polynomials of first and of second order for the exact solution of the IVP.

Due date: Please hand in your homework until Thursday, 24 June, 12:00 into the AM2-box near seminar room Z1, building 01.85 (Fritz-Erler-Str. 1–3).

Tutorial No.9
Advanced Mathematics II

Exercise T25: Determine a solution of the initial value problems applying the power series method. Find an explicit formula for the coefficients.

(a) $(2x - x^2)y''(x) + (1 - x)y'(x) = 0, \quad y(1) = 1, \quad y'(1) = 0$

(b) $(x^2 + 1)u''(x) - 6u(x) = 0, \quad u(0) = 0, \quad u'(0) = 1$

(c) $(x^2 + 1)u''(x) - 6u(x) = 0, \quad u(0) = 1, \quad u'(0) = 0$

Also determine the radius of convergence of the solution in part (c), as well as the general solution of the differential equation $(x^2 + 1)u''(x) - 6u(x) = 0$.

Exercise T26: Use the generalized power series method, i.e., employ the ansatz $y(x) = \sum_{k=0}^{\infty} a_k x^{k+\lambda}$ to determine the general solution of the differential equation

$$x^2 y''(x) + x^2 y'(x) - 2y(x) = 0.$$

Determine the coefficients explicitly.

Exercise T27: Explicitly solve the initial value problem $y'(x) = 2y(x) - 1$, $y(0) = 1$ by means of Euler's method for a step size $h > 0$. Show that for the approximate solution y_k at $k = kh$ one obtains

$$y_k = \frac{(1 + 2h)^k}{2} + \frac{1}{2}.$$

Tutorial date: Friday 18 June