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Karlsruhe, 22 June 2010

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Student Nr.: ....

Worksheet No.10
Advanced Mathematics II

Exercise 46: The gamma function $\Gamma(z)$ is defined by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt, \qquad z \in \mathbb{R}.$$

Show that $\Gamma(z)$ is well-defined for $z > 0$, and that the recursion relation $\Gamma(z+1) = z\Gamma(z)$ holds. Conclude that $\Gamma(n+1) = n!$ for $n = 1, 2, \dots$.

Exercise 47: Determine the parameter integral

$$F(s) = \int_0^\infty f(t) e^{-st} dt = \int_0^\infty e^{3t} e^{-st} dt\,.$$

For which s does the integral exist? Show $\frac{dF(s)}{ds} = \int_0^\infty (-t) e^{3t} e^{-st} dt$, i.e. the derivative of $F(s)$ is the Laplace transform $G(s)$ of $g(t) = -te^{3t} = -tf(t)$. Use Theorem 3.3.

Exercise 48: Determine the Laplace transforms of the following functions by using the computational rules and the table for the Laplace transform:

(a) $f(t) = 3e^{4t} + 2$, (b) $h(t) = e^{-t}\cos(2t)$, (c) $g(t) = \begin{cases} \sin(\omega t - \varphi), & \text{für } \omega t - \varphi \geq 0, \\ 0, & \text{otherwise,} \end{cases}$ for $\omega, \varphi > 0$.

(d) Determine the Laplace transform of the function

$$f(t) = t^2 \sin^2 t, \quad t \in [0, \infty),$$

by means of twice differentiating in the image space. Hint: $\sin^2 t = (1 - \cos 2t)/2$.

Exercise 49: Express the following functions by means of Heaviside's function $H(t) = \begin{cases} 0, & t > 0 \\ 1, & t \geq 0 \end{cases}$:

a) $f(t) = \begin{cases} 1, & t \geq t_0 \\ 0 & \text{otherwise} \end{cases}$ b) $f(t) = \begin{cases} 1, & t \leq t_0 \\ 0, & \text{otherwise} \end{cases}$ c) $f_{\alpha,\beta}(t) = \begin{cases} 1, & \alpha \leq t < \beta \\ 0, & \text{otherwise} \end{cases}$

Determine $\mathcal{L}\left(\frac{1}{\beta-\alpha} f_{\alpha,\beta}\right)$ and its behaviour for $\alpha = 0$ and $\beta \to \alpha$.

Exercise 50: Let $f : [0, \infty) \to \mathbb{R}$ be the function with period 2 that is given in the interval $[0, 2)$ by

$$f(t) = \begin{cases} 1 & \text{for } 0 \leq t < 1, \\ 0 & \text{for } 1 \leq t < 2. \end{cases}$$

(a) Sketch the function's graph in the interval $[0, 10]$.

(b) Compute the Laplace transform of f.

Due date: Please hand in your homework until Thursday, 1 July, 12:00 into the AM2-box near seminar room Z1, building 01.85 (Fritz-Erler-Str. 1–3).

Tutorial No.10
Advanced Mathematics II

Exercise T28: Compute the integral $F(s) := \int_0^\infty f(t)e^{-st}dt$ for $s > 0$ and the functions

(a) $f(x) = 3e^{4x} + 2$ (b) $f(x) = e^{-x}\cos(2x)$.

Exercise T29: Consider the functions

$$H(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases} \qquad \text{and} \qquad f(t) = \begin{cases} t-1, & 1 \leq t < 3, \\ 8-2t, & 3 \leq t < 4, \\ 0, & \text{else.} \end{cases}$$

(a) Rewrite the function f in a form without any case differentiation using the Heaviside function H.

(b) Sketch the graph of the function f and determine its Laplace transform.

Exercise T30: Determine the Laplace transform of:

(a) $f(x) = x^2 + 3x + 4 + x^2\sin(2x)$ (b) $f(x) = \begin{cases} \sin(x), & 0 \leq x < \pi \\ \cos(x), & x \geq \pi \end{cases}$

(c) $f(x) = (e^{2x} + e^{3x}) \cdot \sin(4x)$ (d) $f(x) = \cos(x) - x\sin(x) = (x \cdot \cos(x))'$

(e) $f(x) = x^n$, $n \in \mathbb{N}$

In part (e) use the definition of the Laplace transform and apply mathematical induction.

Tutorial date: Friday 25 June