Karlsruher Institut für Technologie
Institut für Algebra und Geometrie
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Karlsruhe, 16.06.2011

Student Nr.: ....

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Worksheet No.10
Advanced Mathematics II

Exercise 46: The gamma function $\Gamma(z)$ is defined by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt, \qquad z \in \mathbb{R}.$$

Show that $\Gamma(z)$ is well-defined for $z > 0$, and that the recursion relation $\Gamma(z+1) = z\Gamma(z)$ holds. Conclude that $\Gamma(n+1) = n!$ for $n = 1, 2, \ldots$.

Exercise 47: Verify the continuity and differentiability of the parameter integral

$$J(t) = \int_0^1 \arctan(tx)\, dx\,.$$

Compute its derivative for $t \in \mathbb{R}$. Does the limit of the derivative exist for $t \to 0$?

Exercise 48: Determine the Laplace transform of:

(a) $f(x) = x^2 + 3x + 4 + x^2 \sin(2x)$ (b) $f(x) = \begin{cases} \sin(x), & 0 \le x < \pi \\ \cos(x), & x \ge \pi \end{cases}$

(c) $f(x) = (e^{2x} + e^{3x}) \cdot \sin(4x)$ (d) $f(x) = \cos(x) - x\sin(x) = (x \cdot \cos(x))'$

(e) $f(x) = x^n$, $n \in \mathbb{N}$

In part (e) use the definition of the Laplace transform and apply mathematical induction.

Exercise 49: Specify which functions $f(t)$, $t \in [0, \infty)$ have been Laplace-transformed in the given examples:

(a) $F(s) = \dfrac{2s}{s^4 + 2s^3 + 2s^2 + 2s + 1}$, (b) $F(s) = \operatorname{arccot}(s-1)$, (c) $F(s) = \dfrac{e^{-\pi s}}{\sqrt{s^2+1}}$.

Hint: Apply in (a) the partial fraction decomposition. In parts (b) and (c) adopt appropriate computational rules of the Laplace-transform.

Exercise 50: Let the initial value problem $y'(x) = 2y(x) - 1$, $y(0) = 1$ be given.

(a) Compute the exact solution $y(x)$ of the initial value problem.

(b) Compute Euler's method for the initial value problem and step size $h > 0$. Show that for the approximate solution y_k at $x_k = kh$ one obtains $y_k = \frac{(1+2h)^k}{2} + \frac{1}{2}$.

(c) Show that the deviation $|y(x_k) - y_k|$ converges to zero for $h \to 0$.

Due date: Please hand in your homework until Friday, June 24, 12:00 into the AM2-box near seminar room Z1, building 01.85 (Fritz-Erler-Str. 1–3).

Tutorial 10
Advanced Mathematics

Exercise T28: Compute the integral $F(s) := \int\limits_0^\infty f(t)e^{-st}dt$ for $s > 0$ and the functions

$$\text{(a) } f(x) = 3e^{4x} + 2 \qquad \text{(b) } f(x) = e^{-x}\cos(2x).$$

Exercise T29: Determine the Laplace transforms of the following functions by using the computational rules and the table for the Laplace transform:

(a) $f(t) = 3e^{4t} + 2$, (b) $h(t) = e^{-t}\cos(2t)$, (c) $g(t) = \begin{cases} \sin(\omega t - \varphi), & \text{für } \omega t - \varphi \geq 0, \\ 0, & \text{otherwise,} \end{cases}$ for $\omega, \varphi > 0$.

(d) Determine the Laplace transform of the function

$$f(t) = t^2 \sin^2 t, \quad t \in [0, \infty),$$

by means of twice differentiating in the image space. Hint: $\sin^2 t = (1 - \cos 2t)/2$.

Exercise T30: Specify which functions $f(t)$ have been Laplace-transformed in the given examples:

$$\text{(a) } F(s) = \frac{1}{s^5} \qquad \text{(b) } F(s) = \frac{6}{(s-2)^4} \qquad \text{(c) } F(s) = \frac{1}{s(s+1)^2} \qquad \text{(d) } F(s) = \frac{2s}{(s+1)^2(s^2+1)}.$$

For detailed information regarding this course please check the page
`http://www.math.kit.edu/iag1/lehre/am22011s/en`

Tutorial date: Tuesday, June 21.