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Group

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Exercise sheet 6
Advanced Mathematics III
for Mechanical Engineering

Question 1: A satellite of mass $m = 300$ [kg] shall be placed into a geostationary orbit along a path C parametrised by

$$x(t) = \begin{pmatrix} 0 \\ (1+6t)\sin(2\pi t) \\ (1+6t)\cos(2\pi t) \end{pmatrix} \cdot 6 \cdot 10^6 \text{ [m]} , \quad t \in [0,1] .$$

(Let the centre of Earth constitute the origin of the coordinates system to be used.) In doing so, one must act against Earth's gravitational force $F(x) = -GMm\frac{x}{\|x\|^3}$ (Earth's mass: $M = 6 \cdot 10^{24}$ [kg], gravitational constant: $G = 7 \cdot 10^{-11} \left[\frac{N \cdot \text{m}^2}{\text{kg}^2}\right]$). Evaluate the work done $W = -\int_C F(x) \cdot ds$ thus required.

Hint: Does there exist a scalar potential for $-F$? Check the necessary integrability conditions: $\frac{\partial}{\partial x_j}(-F_i) = \frac{\partial}{\partial x_i}(-F_j)$ for $i,j = 1,2,3$. Hereby F_i denotes the i^{th} component of F.

Question 2: A Van–de–Graaff generator is a device to generate large electrostatic potential differences by transporting positive electric charges onto a (eventually insulated) hollow metal sphere. The electric charge on the sphere Q is given by $Q = \delta A$, where A denotes the surface area of the sphere and δ its surface charge density. Determine the amount of electric charge on a spherical shell of the form $x_1^2 + x_2^2 + x_3^2 = 1$ [m], from which a cylinder of the form $x_1^2 + x_2^2 = 1/16$ [m] has cut out a hole in the half space $\{x_3 \le 0\}$. For the surface charge density use $\delta = 1.2 \cdot 10^{-5}$ coulomb/m^2.

Question 3: The parametrisation

$$X(r,\varphi) := (r\cos(\varphi), r\sin(\varphi), \varphi) , \quad 0 \le r \le 1 , \quad 0 \le \varphi \le 2\pi ,$$

defines a surface S. [With no marks given, draw a rough sketch of S.]

a) Determine the equation of the tangential plane T and the unit normal vector of S in the point $(0, 1/2, \pi/2)^\top$.

b) Determine the area of S, and evaluate the surface integral

$$J = \iint_S \sqrt{x_1^2 + x_2^2 + 1}\, do .$$

Question 4: Consider the mantle of a cone described by $M = \left\{(x,y,z)^\top \in \mathbb{R}^3 : x^2+y^2 \le 1,\ z = \sqrt{x^2+y^2}\right\}$.

a) Determine the surface area of M by using Cartesian coordinates.

b) Determine the surface area of M by employing cylindrical polar coordinates.

c) By means of the coordinates introduced in b), evaluate the (outward oriented) flux of the vector field $f(x,y,z) = (x+y, y-x, z^2)^\top$ through M.

Question 5: Consider the vector field $f(x,y,z) = (x,y,1)^\top$ and the body

$$K = \left\{(x,y,z)^\top \in \mathbb{R}^3 : x^2+y^2 \le 2,\ 0 \le z \le x+y+3\right\} .$$

a) Sketch the body K.

b) Give parametrisations of the three smooth surfaces F_1, F_2 and F_3, which form the boundary of K.

c) Evaluate the volume integral $\int_K \operatorname{div} f \, d(x,y,z)$, as well as the flux of f through the surfaces F_1, F_2 and F_3.

Hint: Note that $\operatorname{div} f = \dfrac{\partial f_1}{\partial x_1} + \dfrac{\partial f_2}{\partial x_2} + \dfrac{\partial f_3}{\partial x_3}$ holds.

Deadline: Thursday, December 2, 2010 at 15:45 h