Karlsruher Institut für Technologie (KIT)
Institut für Algebra und Geometrie
PD Dr. F. Hettlich
Dipl.-Math. M. Mitschele

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Group

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Name: ....

Matrikel-Nr.: ....

Exercise sheet 7
Advanced Mathematics III
for Mechanical Engineering

Question 1: Let $a > 0$. The set $L = \{(x,y)^\top \in \mathbb{R}^2 : (x^2+y^2)^2 = a^2(x^2-y^2)\}$ defines implicitly an algebraic curve in $\mathbb{R}^2$, the so-called *lemniscate*. The two wings of the lemniscate may be parametrised by

$$z_1(t) = a\begin{pmatrix} \sqrt{\cos(2t)}\cos(t) \\ \sqrt{\cos(2t)}\sin(t) \end{pmatrix} \text{ for } t \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right] \quad \text{resp.} \quad z_2(t) = a\begin{pmatrix} \sqrt{\cos(2t)}\cos(t) \\ \sqrt{\cos(2t)}\sin(t) \end{pmatrix} \text{ for } t \in \left[\frac{3\pi}{4}, \frac{5\pi}{4}\right] .$$

a) Sketch the lemniscate.

b) Show that the parametrisation z_1 of the right wing of the lemniscate is smooth.

c) Compute by means of Gauß' integral theorem the total area D which is enclosed by the lemniscate. For this purpose consider the function $f(x,y) = (x,0)^\top$.

Question 2: Consider the vector field $f(x,y,z) = (y(z-1),\, 0,\, z-1)^\top$. Determine the flux of f through the surface of the semisphere

$$H = \left\{(x,y,z)^\top \in \mathbb{R}^3 : (x-2)^2 + y^2 + (z-1)^2 \le 1,\ 1 \le z\right\} .$$

a) Find parametrisations for the two smooth partial surfaces F_1 and F_2 which form the boundary of H.

b) Evaluate $\int_H \operatorname{div} f \, d(x,y,z)$, as well as the flux of f through each of the partial surfaces F_1 and F_2. Verify in this way Gauß' integral theorem in $\mathbb{R}^3$.

Question 3: Let the restrictions $x_1 < 0$, $x_3 < 2$ and $x_1^2 + x_2^2 + 1 < x_3$ describe a bounded domain $D \subseteq \mathbb{R}^3$. Moreover, let a vector field $F : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$F(x) = \left(e^{x_2 x_3},\ x_1 x_2 x_3,\ e^{x_1+x_2}\right)^\top .$$

Evaluate the surface integral

$$\iint_{\partial D} F \cdot \nu \, do$$

with respect to the boundary ∂D of the domain D, where ν denotes the outward-oriented unit normal vector to ∂D.

Question 4: Let the scalar-valued function $f(x) = \frac{1}{2}\ln\left(x_1^2 + x_2^2\right)$ be defined on the annulus $D : r \le \|x\|_2 \le 1$, for some $r \in (0,1)$. Let ν denote the outward-pointing unit normal vector to ∂D. Evaluate the line integral

$$\int_{\partial D} \frac{\partial f}{\partial \nu}(x)\, ds$$

(i) directly, and (ii) by means of Green's first integral formula.

Question 5: The algebraic equation $z = x^3 - y^3 - x + 1$ defines a surface in $\mathbb{R}^3$, the normal vector of which be upward-oriented (i.e., the z-coordinate of the normal vector be positive). Consider the part of this surface which is given by

$$S = \{(x,y,z) \in \mathbb{R}^3 : z = x^3 - y^3 - x + 1,\ 1 < x^2 + y^2 < 4\} .$$

Compute the integral $\int_{\partial S} F \cdot ds$ for the vector field $F(x,y,z) = (2y, z, -1)^\top$ employing Stokes' integral theorem. The boundary curve ∂S of S be oriented in such a way that Stokes' integral theorem may be applied immediately.

Deadline: Thursday, December 9, 2010 at 15:45 h