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Group

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Name: ....

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Exercise sheet 8
Advanced Mathematics III
for Mechanical Engineering

Question 1: Consider the velocity field $u(x,y,z) = (z-y,\ x-z, y-x)^\top$ of a turbulent flow, and a surface given by

$$F = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 :\ x^2 + y^2 \leq 4,\ z = xy \right\}.$$

a) Sketch the surface F.

b) Evaluate on F the integral over all vortex strengths of u, $\iint\limits_F \operatorname{rot} u \cdot do$.

c) Compute the circulation of u, $\int\limits_{\partial F} u \cdot ds$, along the boundary curve ∂F, and verify thus Stokes' integral theorem in $\mathbb{R}^3$.

Question 2: Let a scalar-valued function $u : \mathbb{R}^3 \to \mathbb{R}$ be twice continuously differentiable, and let another scalar-valued function v be defined by

$$v(r, \varphi, z) = u(r\cos\varphi, r\sin\varphi, z)\ .$$

(a) Express the partial derivatives $\frac{\partial v}{\partial r}$, $\frac{\partial v}{\partial \varphi}$ resp. $\frac{\partial v}{\partial z}$ in terms of the corresponding partial derivatives of u.

(b) Show that for $(x,y,z)^\top = (r\cos\varphi, r\sin\varphi, z)^\top$ the transformation relation

$$\frac{\partial^2 u}{\partial x^2}(x,y,z) + \frac{\partial^2 u}{\partial y^2}(x,y,z) + \frac{\partial^2 u}{\partial z^2}(x,y,z) = \frac{\partial^2 v}{\partial r^2}(r,\varphi,z) + \frac{1}{r}\frac{\partial v}{\partial r}(r,\varphi,z) + \frac{1}{r^2}\frac{\partial^2 v}{\partial \varphi^2}(r,\varphi,z) + \frac{\partial^2 v}{\partial z^2}(r,\varphi,z)$$

holds. (This gives the representation of the Laplacian operator in terms of cylindrical polar coordinates.)

Question 3:

(a) Check whether the following partial differential equations of first order classify as linear, quasi-linear, or neither:
i) $(3x+7y)u_x + \cos(xy)u_y + 5e^x u = \sin(x^2)$,
ii) $(2\sin(x) + e^{x+u}\cos(y))u_x + (1-y)(2+x)^u u_y + \sin(3u) = 3$,
iii) $(5+2u)\cos(u_x) + 3u_y + 7 = x^2 + y^2$.

(b) Determine the type of the following partial differential equations of second order, and sketch, when necessary, in $\mathbb{R}^2$ resp. $\mathbb{R}^3$ the domains of different type for a given differential equation:
i) $u_{xx} + 2\sqrt{50}u_{xy} - 4u_{yy} + 3u_x - 7u_y = 2u + 5$,
ii) $2x^2 u_{xx} + 2xyu_{xy} + 2u_{yy} = u + x^2 + y^2 + 1$,
iii) $7u_{xx} + 7u_{yy} - 2u_{zz} - 10u_{xy} + 8u_{xz} + 8u_{yz} + 4\sin(y)u_x - 9u_y - 3xu_z = 0$.

Question 4: Determine the solution u of the intial value problem

$$-\frac{x_1^2}{u(x_1,x_2)}\frac{\partial u(x_1,x_2)}{\partial x_1} + (x_2 - 1)\frac{\partial u(x_1,x_2)}{\partial x_2} + (u(x_1,x_2))^2 = 0\ , \qquad x_1 > 0\ ,\ x_2 \in \mathbb{R}\ ,$$

with

$$u(2, x_2) = 1 \ \text{ for all } x_2 \in \mathbb{R}\ .$$

Question 5: Use the method of characteristics to find the solution u of the initial value problem

$$xu_x(x,y) + \frac{x}{yu(x,y)}u_y(x,y) + u(x,y) = 0\ , \quad x,y > 0\ , \quad u(t^2, t) = 1\ ,\ t > 0\ .$$

Deadline: Thursday, December 16, 2010 at 15:45 h